QUARTERLY

OF

APPLIED MATHEMATICS

VOLUME XI JULY · 1953 NUMBER 2

QUARTERLY

OF

APPLIED MATHEMATICS

This periodical is published quarterly by Brown University, Providence 12, R. I. For its support, an operational fund is being set up to which industrial organizations may contribute. To date, contributions of the following industrial companies are gratefully acknowledged:

BELL TELEPHONE LABORATORIES, INC.; NEW YORK, N. Y.,
THE BRISTOL COMPANY; WATERBURY, CONN.,
CURTISS WRIGHT CORPORATION; AIRPLANE DIVISION; BUFFALO, N. Y.,
EASTMAN KODAK COMPANY; ROCHESTER, N. Y.,
GENERAL ELECTRIC COMPANY; SCHENECTADY, N. Y.,
GULF RESEARCH AND DEVELOPMENT COMPANY; PITTSBURGH, PA.,
LEEDS & NORTHRUP COMPANY; PHILADELPHIA, PA.,
PRATT & WHITNEY, DIVISION NILES-BEMENT-POND COMPANY; WEST HARTFORD, CONN.,
REPUBLIC AVIATION CORPORATION; FARMINGDALE, LONG ISLAND, N. Y.,
UNITED AIRCRAFT CORPORATION; EAST HARTFORD, CONN.,
WESTINGHOUSE ELECTRIC AND MANUFACTURING COMPANY; PITTSBURGH, PA.

The QUARTERLY prints original papers in applied mathematics which have an intimate connection with application in industry or practical science. It is expected that each paper will be of a high scientific standard; that the presentation will be of such character that the paper can be easily read by those to whom it would be of interest; and that the mathematical argument, judged by the standard of the field of application, will be of an advanced character.

Manuscripts submitted for publication in the QUARTERLY OF APPLIED MATHEMATICS should be sent to Professor W. Prager, or Professor G. F. Carrier, Quarterly of Applied Mathematics, Brown University, Providence 12, R. I., either directly or through any one of the Editors or Collaborators. In accordance with their general policy, the Editors welcome particularly contributions which will be of interest both to mathematicians and to engineers. Authors will receive galley proofs only. The authors' institutions will be requested to pay a publication charge of $5.00 per page which, if honored, entitles them to 100 free reprints. Instructions will be sent with galley proofs.

The subscription price for the QUARTERLY is $6.00 per volume (April-January), single copies $2.00. Subscriptions and orders for single copies may be addressed to: Quarterly of Applied Mathematics, Brown University, Providence 12, R. I., or to Box 2-W, Richmond, Va.

Entered as second class matter March 14, 1944, at the post office at Providence, Rhode Island, under the act of March 3, 1879. Additional entry at Richmond, Virginia.

WILLIAM BYRD PRESS, INC., RICHMOND, VIRGINIA

QUARTERLY OF APPLIED MATHEMATICS

Vol. XI July, 1953 No. 2

LINEAR APPROXIMATIONS IN A CLASS OF NON-LINEAR VECTOR DIFFERENTIAL EQUATIONS*

BY

J. J. GILVARRY

The RAND Corporation, Santa Monica, California

1. Introduction. The class of vector differential equations considered in this paper is represented by

$$d^2\mathbf{r}/dt^2 - \mathbf{f}(\mathbf{r}) = \mathbf{b}(t), \tag{1}$$

in which $\mathbf{r}$ is a radius vector in a Cartesian frame, $\mathbf{f}(\mathbf{r})$ is a point function derivable as the gradient of a scalar, and $\mathbf{b}(t)$ is an arbitrary function of the independent variable t. In its physical application, Eq. (1) determines the position of a particle subject to the acceleration $\mathbf{b} + \mathbf{f}$ at time t. To allow the wider interpretation of Eq. (1) as a curve in three-space (for fixed initial conditions), t will not be identified with time (except in the applications of Sec. 3). The basic assumption will be made that $\mathbf{f}(\mathbf{r})$ (which, in general, is a non-linear function of components of $\mathbf{r}$) is a slowly varying function as compared to arc length on any integral curve of Eq. (1). The scalar corresponding to $\mathbf{f}(\mathbf{r})$ will be assumed differentiable to fourth order at least, and the function $\mathbf{b}(t)$ will be assumed differentiable at the initial point of any integral curve of Eq. (1) to third order at least (but otherwise merely Riemann-integrable).

In a previous paper (referred to hereafter as *I*) by the author and others [1], a special case of Eq. (1) was treated, in which $\mathbf{f}(\mathbf{r})$ and $\mathbf{b}(t)$ were the gravitational and non-gravitational accelerations respectively of a particle. A quasi-linear approximation for two-dimentional motion was described, which takes into account terms in $\mathbf{f}$ of higher order than does a linearized approximation, and which depends on the assumption that the radius of curvature of the corresponding trajectory be slowly varying. In this paper, the quasi-linear approximation in question is generalized in two directions: (a) to three dimensions (by assuming that the radius of torsion of an integral curve is likewise slowly varying); (b) to include an arbitrary $\mathbf{f}(\mathbf{r})$ derivable as the gradient of a non-harmonic scalar. The approximation exploits the fact that coordinate distances (canonical coordinates) along the tangent, principal normal, and binormal of an analytic integral curve are proportional (within dominant terms) to the first, second, and third powers of the arc length, respectively, by virtue of the Frenet-Serret formulas. Hence, non-linear terms in $\mathbf{f}$ through third order in the arc length can be expressed as linear functions of the canonical coordinates and thus of any Cartesian coordinates. In a sense, the approximation considered is the analogue in the non-linear case of Liouville's approxi-

*Received June 9, 1952.

mation [2, 3] (referred to as the Wentzel-Kramers-Brillouin-Jeffries approximation [4, 5, 6] in quantum mechanics). A prefatory discussion of the linearized approximation to Eq. (1) is given first.

Some remarks on the practical use of approximations of the type considered are in order. The general function of such approximations is to determine corrections to a zero-or first-order solution for a limited range of the independent variable. Approximations of this type cannot compete, of course, with modern numerical methods of solution for an extended range of the independent variable. For moderate ranges of the independent variable, however, linear approximations can be applied piece-wise by identifying initial conditions on one interval with terminal conditions on a preceding interval. Such a continuation solution has been used in I for the computation of an actual trajectory.

2. Linearized Approximation. The scalar potential φ, in terms of which $\mathbf{f} = \nabla\varphi$, will be expressed as a function of three coordinates x_i $(i = 1,2,3)$ forming a Cartesian frame (right-handed in the order given) with origin at the initial point O of an integral curve in question. The Taylor expansion of φ in these coordinates at O will have the form

$$\varphi = \varphi^{(1)} + \varphi^{(2)} + \varphi^{(3)} + \varphi^{(4)} + \cdots, \tag{2}$$

where $\varphi = 0$ at O and each term is a rational integral homogeneous function of the x_i (the superscript notation indicates the degree). Through quadratic terms, φ will be written

$$\varphi = \sum A_i x_i + \sum B_{ij} x_i x_j, \qquad (i, j = 1, 2, 3), \tag{3}$$

where $B_{ij} = B_{ji}$.

A matrix notation $\{a\}$ will be used interchangeably with $\mathbf{a}$ for a vector, where $\{a\}$ stands for the column vector $\{a_1, a_2, a_3\}$ of components. The linearized approximation $\mathbf{f}^{(1)}$ to $\mathbf{f}$ consists in retaining only terms in φ through second order. From Eq. (3), it is

$$\{f^{(1)}\} = \mathbf{A} + \mathbf{P}\{r\}, \tag{4}$$

where $\{r\} = \{x_i\}$, $\mathbf{A} = \{A_i\}$, and the coefficient matrix $\mathbf{P}$ is

$$\mathbf{P} = 2\mathbf{B} \tag{5}$$

in terms of the matrix $\mathbf{B} \equiv [B_{ij}]$. With the notation $\mathbf{r}^{\cdot} = d\mathbf{r}/dt$, the corresponding linearized approximation to Eq. (1) is

$$\{r^{\cdot\cdot}\} - \mathbf{P}\{r\} = \{b\} + \mathbf{A}, \tag{6}$$

which assumes that $\mathbf{f}$ is a slowly-varying function of position with respect to arc length on an integral curve.

Since $\mathbf{P}$ is a symmetric matrix, its eigenvalues are real, and Eq. (6) can be solved directly by reduction to normal coordinates. The solution for the normal coordinate x_i' corresponding to the negative eigenvalue $-\omega_i^2$ of $\mathbf{P}$ is

$$x_i' = v_i'\omega_i^{-1}\sin\omega_i t + \omega_i^{-1}\int_0^t [b_i'(\tau) + A_i']\sin\omega_i(t-\tau)\,d\tau, \tag{7}$$

where v_i' is the initial value of dx_i'/dt and $\mathbf{b}' + \mathbf{A}'$ is the transform in the normal coordinates of $\mathbf{b} + \mathbf{A}$. For a positive eigenvalue of $\mathbf{P}$, the circular functions in Eq. (7) must

be replaced by the corresponding hyperbolic functions. In the classical theory of small oscillations of a particle, eigenvalues leading to such hyperbolic solutions are usually excluded. Such exclusion is not necessary (at least on purely mathematical grounds) in the application to particle motion of the linear approximations discussed here, since the application is to the initial phase of a forced motion. For this reason, the approximations do not necessarily carry implications on the boundedness in the large of the corresponding motion. Note that, from Earnshaw's theorem [7], all the eigenvalues of **P** can be of one sign only if φ is non-harmonic.

Ordinarily, the coefficients B_{ij} must be determined by direct expansion of φ. The Appendix describes a method of determining these coefficients which relates them to geometric parameters of the equipotential passing through the origin O. For the purpose of formulating this result, take the x_3-axis in the direction of $-\nabla\varphi$, and take the x_1-, x_2- axes in the directions of the lines of curvature of the equipotential surface $\varphi = 0$ in the tangent plane of this equipotential at O. Let r_1 and r_2 be the principal radii of normal curvature of this surface at O in the directions of x_1 and x_2 respectively[1]. Let $r_{1,2}$ be the derivative evaluated at O of the principal radius of normal curvature in the x_1-direction with respect to arc length on the line of curvature in the x_2-direction; and let $r_{2,1}$ be defined correspondingly. The matrix **B** is then

$$\mathbf{B} = \frac{f_0}{2}\begin{bmatrix} r_1^{-1} & 0 & -r_{2,1}r_2^{-1} \\ 0 & r_2^{-1} & -r_{1,2}r_1^{-1} \\ -r_{2,1}r_2^{-1} & -r_{1,2}r_1^{-1} & -(r_1^{-1} + r_2^{-1} + \sigma_0/f_0) \end{bmatrix}, \tag{8}$$

where σ_0 is a mass density defined by Poisson's equation $\nabla^2\varphi = -\sigma$ at O, and $f_0 = |\nabla\varphi|$ at O. The vector **A** is $\{0, 0, -f_0\}$ in the particular frame introduced here. Equation (8) is convenient when the form of the equipotential is a datum given directly (as it is for **f** a gravitational acceleration). Use will be made of it in an application in Sec. 3.

3. Quasi-Linear Approximation. For a non-vanishing initial $d\mathbf{r}/dt$, the corresponding integral curve is analytic at the initial point O, since $\nabla\varphi$ and **b** are assumed differentiable at O (to third order, at least). Hence, the radius vector **r** to a point on the curve can be expanded in the Taylor series [8]

$$\mathbf{r} = \boldsymbol{\alpha}s + \frac{\boldsymbol{\beta}}{2\rho}s^2 - \left\{\frac{\boldsymbol{\alpha}}{\rho^2} + \frac{\rho'\boldsymbol{\beta}}{\rho^2} + \frac{\boldsymbol{\gamma}}{\rho\tau}\right\}\frac{s^3}{6} + \cdots, \tag{9}$$

where s is the arc length on the curve; $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ are the unit tangent, unit principal normal, and unit binormal at O respectively; and ρ, τ, ρ' are the (non-vanishing) radius of curvature, (non-vanishing) radius of torsion, and derivative of ρ with respect to s respectively on the integral curve at O. Equation (9) is equivalent to the standard Frenet-Serret formulas. It is understood that $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, ρ, τ, ρ' are constants evaluated at O, but specifying subscripts will be omitted for convenience.

The coefficients of the terms of order s^4 and higher in Eq. (9) contain products of powers and derivatives with respect to s of $1/\rho$ and $1/\tau$. If all derivatives of ρ and τ with respect to s are sufficiently small, one can take s sufficiently small so that terms

[1]The principal radii r_1 and r_2 are taken positive or negative according as the corresponding centers of curvature lie on the positive or negative half of the x_3-axis.

of the form s^4/ρ^3, $s^4/\rho^2\tau$, $\cdots$ (which do not involve derivatives) are negligible in Eq. (9). In this case, $\mathbf{r}$ is given by the terms of the series (9) which are indicated explicitly. The permissible range of s for this approximation is given by $s \ll \rho$, τ and is thus smaller as ρ and τ are smaller. Correspondingly, the terms of order s^2 and s^3 in Eq. (9) are then smaller for ρ and τ smaller. Since these terms form the basis of the quasi-linear approximation to be described, the range of s over which this approximation yields a significant correction to the linearized approximation is greater for ρ and τ larger.

On these assumptions, Eq. (9) determines any x_i as a linear function of s, s^2 and s^3, which is given by

$$\{r\} = \mathbf{\Gamma}'\mathbf{S}\{s^i\}, \tag{10}$$

where $\{s^i\}$ designates the column vector $\{s, s^2, s^3\}$, $\mathbf{\Gamma}'$ is the transpose of the matrix $\mathbf{\Gamma}$ of direction cosines of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$,

$$\mathbf{\Gamma} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}, \tag{11}$$

and the matrix $\mathbf{S}$ is

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & -(6\rho^2)^{-1} \\ 0 & (2\rho)^{-1} & -\rho'(6\rho^2)^{-1} \\ 0 & 0 & -(6\rho\tau)^{-1} \end{bmatrix}. \tag{12}$$

Note that the components of the column vector $\mathbf{S}\ \{s^i\}$ in Eq. (10) are simply canonical coordinates of the curve. Equation (10) can be inverted and written

$$\{s^i\} = \mathbf{S}^{-1}\mathbf{\Gamma}\{r\}, \tag{13}$$

where

$$\mathbf{S}^{-1} = \begin{bmatrix} 1 & 0 & -\tau\rho^{-1} \\ 0 & 2\rho & -2\rho'\tau \\ 0 & 0 & -6\rho\tau \end{bmatrix}, \tag{14}$$

provided $\mathbf{S}$ is non-singular, which is true if the determinant

$$|\ \mathbf{S}\ | = -(12\rho^2\tau)^{-1} \tag{15}$$

be non-vanishing. Thus the following results apply to an integral curve of Eq. (1) which is curved ($1/\rho \neq 0$) and twisted ($1/\tau \neq 0$) at the origin; the special case of an integral curve which is plane at the origin will be treated later. Under these restrictions, Eq. (13) expresses s, s^2 and s^3 linearly in terms of coordinates x_i of the curve.

If quintic and higher terms in φ are neglected, $\mathbf{f}$ can be written

$$\{f\} = \mathbf{A} + \mathbf{P}\{r\} + \{\nabla\varphi^{(3)}\} + \{\nabla\varphi^{(4)}\}. \tag{16}$$

In general, the cubic and quartic terms of φ, $\varphi^{(3)}$ and $\varphi^{(4)}$, are polynomials having respectively 10 and 15 coefficients (these two sets of coefficients can depend on only 7 and 9 independent constants respectively if φ is harmonic) [9]. These terms will be written

$$\varphi^{(3)} = \sum C_{ijk}x_ix_jx_k\,, \qquad (i, j, k = 1, 2, 3), \tag{17a}$$

$$\varphi^{(4)} = \sum D_{ijkl}x_ix_jx_kx_l\,, \qquad (i, j, k, l = 1, 2, 3), \tag{17b}$$

where the indices run over the ranges given to yield no redundancy of symmetric terms (differing only in subscript order in the coefficients C_{ijk}, D_{ijkl}). Thus, the components of $\Delta\,\varphi^{(3)}$ and $\Delta\,\varphi^{(4)}$ are homogeneous polynomials of degree two and three respectively in x_i. But, from Eq. (10), any power x_i^q $(1 \leq q \leq 3)$ can be expressed as a polynomial in s of lowest term s^q. If the expressions derived in this way for x_i^q are substituted in the expressions for $\nabla\,\varphi^{(3)}$ and $\nabla\,\varphi^{(4)}$ corresponding to Eqs. (17a) and (17b) respectively, and powers of s above the third are dropped, one obtains

$$\{f\} = \mathbf{A} + \mathbf{P}\{r\} + (\mathbf{E} + \mathbf{F})\{s^i\}, \tag{18}$$

where the square matrices $\mathbf{E}$ and $\mathbf{F}$ are derived from the terms $\nabla\,\varphi^{(3)}$ and $\nabla\,\varphi^{(4)}$ respectively of $\mathbf{f}$ in Eq. (16). Since $\nabla\,\varphi^{(3)}$ is quadratic in x_i, it can contribute no term to $\mathbf{f}$ which is linear in s; hence, $\mathbf{E} \equiv [E_{ij}]$ can be partitioned by columns thus

$$\mathbf{E} = [0, E_{i2}\,, E_{i3}]. \tag{19}$$

Similarly, $\nabla\,\varphi^{(4)}$ can contribute no terms to $\mathbf{f}$ which are linear or quadratic in s; hence, $\mathbf{F} \equiv [F_{ij}]$ can be partitioned

$$\mathbf{F} = [0, 0, F_{i3}]. \tag{20}$$

The non-vanishing elements of $\mathbf{E}$ and $\mathbf{F}$ are given by

$$E_{i2} = 3C_{iii}\alpha_i^2 + 2\alpha_i \sum_p C_{iip}\alpha_p + \sum_{p,q} C_{ipq}\alpha_p\alpha_q\,, \tag{21a}$$

$$E_{i3} = \rho^{-1}\left[3C_{iii}\alpha_i\beta_i + \sum_p C_{iip}(\alpha_i\beta_p + \beta_i\alpha_p) + \frac{1}{2}\sum_{p,q} C_{ipq}(\alpha_p\beta_q + \beta_p\alpha_q)\right], \tag{21b}$$

$$F_{i3} = 4D_{iiii}\alpha_i^3 + 3\alpha_i^2 \sum_p D_{iiip}\alpha_p + 2\alpha_i \sum_{p,q} D_{iipq}\alpha_p\alpha_q + \sum_{p,q,r} D_{ipqr}\alpha_p\alpha_q\alpha_r\,, \tag{21c}$$

where the indices p, q, r run over the range 1, 2, 3 exclusive of i to yield no redundancy of terms differing only in order of subscripts in the coefficients C_{ijk}, D_{ijkl}. Equation (18) contains the non-linear terms of $\nabla\,\varphi$ through order s^3 expressed as a linear function of s^2 and s^3.

Finally, by means of the linear relationship (13) between $\{s^i\}$ and $\{r\}$, Eq. (18) yields the quasi-linear approximation $\mathbf{f}^{(2)}$ to $\mathbf{f}$,

$$\{f^{(2)}\} = \mathbf{A} + (\mathbf{P} + \mathbf{Q})\{r\}, \tag{22}$$

where

$$\mathbf{Q} = (\mathbf{E} + \mathbf{F})\mathbf{S}^{-1}\mathbf{\Gamma} \tag{23}$$

is a constant square matrix whose elements depend only on coefficients of φ and the initial conditions of the motion (through the parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, ρ, τ, ρ'). The parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ entering $\mathbf{Q}$ through $\boldsymbol{\Gamma}$ can be evaluated directly from standard formulas [8] as

$$\boldsymbol{\alpha} = \frac{\mathbf{v}_0}{v_0}, \qquad \boldsymbol{\beta} = \frac{\mathbf{v}_0 \times (\mathbf{b}_0 + \mathbf{A})}{|\, \mathbf{v}_0 \times (\mathbf{b}_0 + \mathbf{A})\, |} \times \boldsymbol{\alpha}, \qquad \boldsymbol{\gamma} = \boldsymbol{\alpha} \times \boldsymbol{\beta}, \tag{24}$$

where $\mathbf{v}_0$ (of magnitude $v_0 \neq 0$) is the initial value of $d\mathbf{r}/dt$, and $\mathbf{b}_0 \neq -\mathbf{A}$ is the initial value of $\mathbf{b}$. The values of ρ, τ and ρ' in $\mathbf{S}^{-1}$ are given by

$$1/\rho = v_0^{-3}\, |\, \mathbf{v}_0 \times (\mathbf{b}_0 + \mathbf{A})\, |, \tag{25a}$$

$$1/\tau = \rho v_0^{-3} \mathbf{v}_0 \times (\mathbf{b}_0 + \mathbf{A}) \cdot \mathbf{d}_0 \tag{25b}$$

$$\rho' = \rho v_0^{-3}[3\mathbf{v}_0 \cdot (\mathbf{b}_0 + \mathbf{A}) - \rho\boldsymbol{\beta} \cdot \mathbf{d}_0], \tag{25c}$$

where the vector $\mathbf{d}_0$ is defined by

$$\{d_0\} = \{(b^{\cdot})_0\} + \mathbf{P}\{v_0\}. \tag{26}$$

The corresponding quasi-linear approximation to the differential equation (1) is

$$\{r^{\cdot\cdot}\} - (\mathbf{P} + \mathbf{Q})\{r\} = \{b\} + \mathbf{A}. \tag{27}$$

The matrix $\mathbf{Q}$ vanishes and hence Eq. (27) reduces to the linearized equation (6) as $\rho, \tau \to 0$; thus the magnitude of the quasi-linear correction as well as the range of s over which it is valid is greater for ρ and τ larger.

The quasi-linear equation (27) differs from the linearized equation (6) in an important respect. Unlike $\mathbf{P}$, the matrix $\mathbf{P} + \mathbf{Q}$ is not symmetric in general, and thus Eq. (27) implies some unilateral coupling of its component equations. Since $\mathbf{P} + \mathbf{Q}$ can have complex eigenvalues, reduction of the equation to normal coordinates is not generally possible. In view of this complication, analytic solution of the system (27) is best carried out by the Laplace transform method or other operational technique.

The quasi-linear approximation (22) for $\mathbf{f}$ is equivalent to the expression of Eq. (18), which clearly contains all terms in $\mathbf{f}$ of order s^3 or lower. Since $s = O(t)$ when $v_0 \neq 0$, it follows that the Taylor expansion of the quasi-linear solution agrees with the true Taylor expansion of $\mathbf{r}$ through terms[2] of order t^5, as compared to the linearized solution, which agrees through terms of order t^3. For v_0 non-vanishing, an integral curve of the quasi-linear equation has contact of fifth order with the actual integral curve at the initial point. The approximation is of osculating type in general, since it amounts to evaluating $\mathbf{f}$ on the osculating sphere of the integral curve at the initial point.

The constraint that an integral curve lie in an equipotential surface requires a condition on $\mathbf{b}$ for a given $\mathbf{f}$; the condition [10] is

$$b_n + f_n = (\mathbf{r}^{\cdot})^2/r_t\,, \tag{28}$$

where $b_n + f_n$ is the component of $\mathbf{b} + \mathbf{f}$ normal to the surface and r_t is the radius of normal curvature of the surface in the direction of the curve. When the potential φ is spherically symmetric, the quasi-linear equation (27) becomes exact for integral curves

[2]The partial sums through terms of order t^5 of both Taylor expansions exist, because of the differentiability assumptions on $\nabla\varphi$ and $\mathbf{b}$.

lying entirely in an equipotential. To show this result, select x_3 in the direction of $-\nabla\varphi$ (of magnitude f_0) at O, and let R_0 be the directed distance from O to the center of symmetry. From the symmetry of the field $\nabla\varphi$, one can write directly

$$\{r^{\cdot\cdot}\} - f_0R_0^{-1}\{r\} = \{b\} + \{0, 0, -f_0\} \tag{29}$$

for an integral curve on the equipotential $\varphi = 0$. For such an integral curve which is twisted at O, the partial sums through terms of order t^5 of the Taylor expansions of $\mathbf{r}$ are identical for the solution of Eq. (29) and the solution of the quasi-linear equation (27). This fact is sufficient to show that $f_0R_0^{-1}$ is a triple eigenvalue of $\mathbf{P} + \mathbf{Q}$, and thus one has

$$\mathbf{P} + \mathbf{Q} = f_0R_0^{-1}\mathbf{I} \tag{30}$$

in this case, where $\mathbf{I}$ is a unit matrix. A corresponding theorem can be proved for the case where φ is cylindrically symmetric; the theorem for plane symmetry is trivial. In these special cases, the matrix $\mathbf{P} + \mathbf{Q}$ is symmetric and all its eigenvalues are of one sign (even if φ is harmonic).

The preceding treatment is not valid for an integral curve which is plane ($1/\tau = 0$) at the initial point O, because the inversion of Eq. (10) to yield Eq. (13) is not possible. In this case, if ρ be finite and non-vanishing at the initial point, an analogous treatment can be carried out by noting that $\mathbf{S}$ has a non-singular submatrix. In the corresponding development, of which the results will be stated, only terms in φ through the cubic $\varphi^{(3)}$ can be included. Let $\mathbf{T}^{-1}$ and $\mathbf{\Delta}$ be the submatrices

$$\mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2\rho \end{bmatrix}, \qquad \mathbf{\Delta} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{bmatrix}, \tag{31}$$

of $\mathbf{S}^{-1}$ and $\mathbf{\Gamma}$ respectively. If one writes

$$\mathbf{R} = [0, E_{i2}]\mathbf{T}^{-1}\mathbf{\Delta}, \tag{32}$$

where the partitioned matrix $[0, E_{i2}]$ is of 3×2 order, the corresponding quasi-linear approximation to Eq. (1) is

$$\{r^{\cdot\cdot}\} - (\mathbf{P} + \mathbf{R})\{r\} = \{b\} + \mathbf{A}, \tag{33}$$

which reduces to the linearized equation (6) for $\rho \to 0$. For v_0 non-vanishing, an integral curve of Eq. (33) has contact of fourth order with the actual integral curve at the initial point, and the corresponding Taylor expansion of $\mathbf{r}$ agrees with the true Taylor expansion through terms of order t^4.

4. Applications. The preceding results will be applied to the case where $\mathbf{f}$ represents a gravitational acceleration. The value of the coefficient matrix $\mathbf{P}$ of the linearized equation which corresponds to a spheroidal earth rotating with angular velocity Ω will be obtained from Eq. (8) on the basis of Clairaut's first-order theory [11] of the variation of gravity. This matrix for the case of a non-rotating spherical earth is given in *I*. A solution of the quasi-linear equation (33) for plane motion corresponding to this value of $\mathbf{P}$ will be obtained.

The initial point O will be taken on the surface of the earth (Fig. 1), with the x_3-axis directed along the outward normal to the surface, the x_1-axis in the direction of a meri-

dian, and the x_2-axis in the direction of a parallel of latitude (with right-hand sense). The potential φ at points external to the earth can be written

$$\varphi = \varphi_a + \varphi_c \, , \tag{34}$$

in which φ_a is due to Newtonian attraction, and

$$\varphi_c = \frac{1}{2} (\Omega R \sin \theta)^2 \tag{35}$$

is the potential due to centrifugal reaction at the point (R, θ), where R is the radial distance from the earth's center and θ is the corresponding polar angle from the earth's axis. The potential φ_a can be represented as due to sources entirely within the earth and thus is harmonic at the earth's surface; hence, the mass density σ (which is fictitious in this case) is fixed by φ_c alone as

$$\sigma = -\Omega^2. \tag{36}$$

On the first-order theory [11], the potential (34) leads to a radius R of the earth for colatitude θ, and a surface value g of the gravitational acceleration given by

$$R = R_{eq}(1 - A \cos^2 \theta), \tag{37a}$$

$$g = g_{eq}(1 + B \cos^2 \theta), \tag{37b}$$

where R_{eq} and g_{eq} are equatorial values of R and g respectively, and A and B are parameters which depend on geometric constants of the earth, its mass, and its angular velocity. The lines of curvature of the (approximate) ellipsoid of revolution represented by Eq. (37a) are the meridians and parallels of latitude; the principal radii r_1 and r_2 of normal curvature at O are equal respectively to the (negative) radius of curvature of the meridian at this point and the (negative) distance from this point to the polar axis along the surface normal [8]. Thus, one can write from Eq. (8),

$$\mathbf{P} = -\omega_{eq}^2 \begin{bmatrix} 1 + P'_{11} & 0 & P'_{13} \\ 0 & 1 + P'_{22} & 0 \\ P'_{31} & 0 & -2 - P'_{33} \end{bmatrix}, \tag{38}$$

where

$$\omega_{eq}^2 = g_{eq}/R_{eq} \, , \tag{39}$$

and the assumption $A, B \ll 1$ yields

$$P'_{11} = 2A + (B - 3A) \cos^2 \theta, \tag{40a}$$

$$P'_{22} = (B - A) \cos^2 \theta, \tag{40b}$$

$$P'_{33} = 2\frac{2}{5} A + \frac{2}{5} B + 2(B - 2A) \cos^2 \theta, \tag{40c}$$

$$P'_{13} = P'_{31} = A \sin 2\theta. \tag{40d}$$

In determining Eq. (40c), use has been made of Clairaut's theorem [11],

$$\frac{\Omega^2}{\omega_{eq}^2} = \frac{2}{5}(A + B), \tag{41}$$

so that the elements of the coefficient matrix **P** do not contain the earth's angular velocity Ω explicitly.

To determine the coefficient matrix **R** of the quasi-linear equation for plane motion $(1/\tau = 0)$ in a gravitational field, it is sufficiently accurate for the purpose at hand to take $\varphi = g_e R_e^2/R$, which corresponds to a non-rotating spherical earth of mean radius R_e and corresponding mean surface gravity g_e (as defined in I). The value of **R** is then

$$\mathbf{R} = \omega_{eq}^2 [R'_{ij}], \tag{42}$$

where, for $j = 1, 2, 3$,

$$R'_{1j} = 6\lambda\alpha_1\alpha_3\beta_j\rho/R_{eq} , \tag{43a}$$

$$R'_{2j} = 6\lambda\alpha_2\alpha_3\beta_j\rho/R_{eq} , \tag{43b}$$

$$R'_{3j} = 3\lambda(1 - 3\alpha_3^2)\beta_j\rho/R_{eq} , \tag{43c}$$

in which the factor

$$\lambda = 1 + \frac{2}{3}A + \frac{1}{3}B \tag{44}$$

arises from expressing g_e and R_e in terms of g_{eq} and R_{eq} respectively by means of Eqs. (37) and results in I. Equations (42) and (43) generalize to an arbitrary plane of motion the corresponding result[3] in I.

A solution of the quasi-linear equation (33) corresponding to **P** of Eq. (38) and **R** of Eq. (42) will be obtained for the special case of motion in a vertical plane at O. Take new axes x_1' , x_3' at O (Fig. 1), where x_1' is along the trace of this vertical plane in the x_1 ,

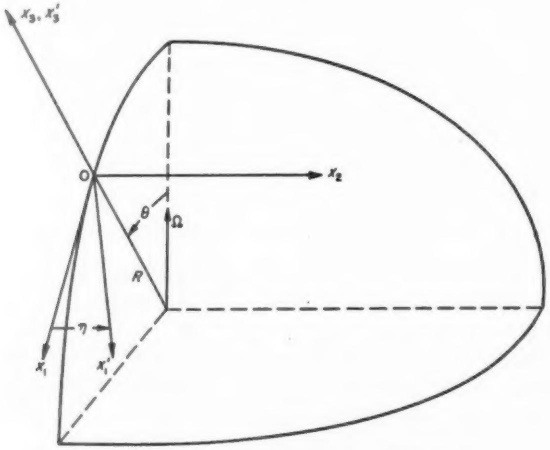

FIG. 1. Coordinate frames on spheroidal earth.

[3]The coefficient matrix $\mathbf{A}_0$ of Eq. (45) in I is in error as printed; the element in the first row and second column should have a negative sign prefixed. Otherwise, the results correspond with the identification x_1 , x_2 , x_3 = x, y, z; $\alpha_2 = \beta_2 = 0$ in Eqs. (42) and (43) above.

x_2 plane and $x_3' = x_3$. Let η be the azimuth (positive sense from x_1 to x_2) of the x_1'-axis relative to the x_1-axis. The transform of Eq. (33) in the x_1', x_3' frame is

$$\begin{Bmatrix} x_1'^{\cdot\cdot} \\ x_3'^{\cdot\cdot} \end{Bmatrix} + \omega_{eq}^2 \begin{bmatrix} 1 + p_{11} & p_{13} \\ p_{31} & -2 - p_{33} \end{bmatrix} \begin{Bmatrix} x_1' \\ x_3' \end{Bmatrix} = \begin{Bmatrix} b_1' \\ b_3' - g \end{Bmatrix}, \tag{45}$$

where

$$p_{11} = P_{11}' \cos^2 \eta + P_{22}' \sin^2 \eta - R_{11}', \tag{46a}$$

$$p_{33} = P_{33}' + R_{33}', \tag{46b}$$

$$p_{13} = P_{13}' \cos \eta - R_{13}', \tag{46c}$$

$$p_{31} = P_{31}' \cos \eta - R_{31}', \tag{46d}$$

in which the R_{ij}' are defined by Eqs. (43) in terms of direction cosines $\boldsymbol{\alpha}'$, $\boldsymbol{\beta}'$ referred to the x_1', x_3' frame. Under the approximation $| p_{ij} | \ll 1$, the solution of the quasi-linear system (45) corresponding to the initial velocity $\{v_1', v_3'\}$ is

$$\begin{aligned} x_1' = v_1'\omega_1^{-1} \sin \omega_1 t + \omega_1^{-1} \int_0^t b_1'(\tau) \sin \omega_1(t - \tau)\, d\tau \\ + \frac{1}{3} p_{13}\left\{v_3's(t) + \int_0^t [b_3'(\tau) - g]s(t - \tau)\, d\tau\right\}, \end{aligned} \tag{47a}$$

$$\begin{aligned} x_3' = v_3'\omega_3^{-1} \sinh \omega_3 t + \omega_3^{-1} \int_0^t [b_3'(\tau) - g] \sinh \omega_3(t - \tau)\, d\tau \\ + \frac{1}{3} p_{31}\left\{v_1's(t) + \int_0^t b_1'(\tau)s(t - \tau)\, d\tau\right\}, \end{aligned} \tag{47b}$$

where

$$\omega_1 = \omega_{eq}\left(1 + \frac{1}{2} p_{11}\right), \qquad \omega_3 = 2^{1/2}\omega_{eq}\left(1 + \frac{1}{4} p_{33}\right), \tag{48}$$

and the function $s(t)$ is defined by

$$s(t) = \omega_1^{-1} \sin \omega_1 t - \omega_3^{-1} \sinh \omega_3 t. \tag{49}$$

The corresponding linearized solution is obtained by letting $\rho \to 0$ in the parameters p_{ij}. Note that the restriction $| p_{ij} | \ll 1$ imposed to obtain the solution (47) requires $\rho \ll R_{eq}$ in Eqs. (43) for the elements of the matrix $\mathbf{R}$.

The author wishes to acknowledge helpful discussions of this problem with Dr. R. Isaacs, Dr. A. Latter, Dr. G. Peebles and Mr. H. Kahn of The RAND Corporation, and with Prof. T. Dantzig of the RAND consulting staff.

Appendix

In this appendix, Eq. (8) of the text for the matrix **B** will be derived. Take coordinate axes as specified by the text in connection with this equation. It is known that, in these coordinates, a plane parallel to the tangent plane of a surface and at the signed distance

$p = x_3$ from it cuts the surface in a curve (the Dupin indicatrix) defined within terms of second order by [8]

$$p = x_1^2/2r_1 + x_2^2/2r_2 + \cdots, \qquad (x_3 = p), \tag{1A}$$

where r_1 and r_2 are the principal radii of normal curvature of the surface at O in the directions of x_1 and x_2, respectively. From Eq. (2), this curve is likewise given by

$$p = f_0^{-1}(B_{11}x_1^2 + 2B_{12}x_1x_2 + B_{22}x_2^2) + \cdots, \qquad (x_3 = p), \tag{2A}$$

if $f_0 = | \nabla \varphi |$ at O. Comparison of Eqs. (1A) and (2A) yields

$$B_{11} = \frac{1}{2} f_0 r_1^{-1}, \qquad B_{12} = 0, \qquad B_{22} = \frac{1}{2} f_0 r_2^{-1}. \tag{3A}$$

The coefficient B_{33} can be fixed by the condition that φ, in general, satisfies Poisson's equation $\nabla^2 \varphi = -\sigma$, where σ is a mass density. This condition yields

$$B_{33} = -\frac{1}{2} (f_0 K_m + \sigma_0), \tag{4A}$$

where K_m is the mean curvature $r_1^{-1} + r_2^{-1}$ of the equipotential and σ_0 is the mass density at O.

To evaluate B_{13}, consider a plane normal to the x_1-axis at the signed distance $p = x_1$ from O. One can show that the equation of the curve of intersection of this plane with the equipotential $\varphi = 0$ is, within second-order terms [8],

$$x_3(1 + r_{2,1}p/r_2) = p^2/2r_1 + x_2^2/2r_2 + \cdots, \qquad (x_1 = p), \tag{5A}$$

where $r_{2,1}$ is the derivative evaluated at O of the principal radius of normal curvature in the x_2-direction with respect to arc length on the line of curvature in the x_1-direction. From Eqs. (2) and (3A), this curve is likewise defined by

$$x_3(1 - 2f_0^{-1}B_{13}p) = p^2/2r_1 + x_2^2/2r_2 + \cdots, \qquad (x_1 = p), \tag{6A}$$

which yields

$$B_{13} = -\frac{1}{2} f_0 r_{2,1} r_2^{-1} \tag{7A}$$

by comparison with Eq. (5A). Similarly, one has

$$B_{23} = -\frac{1}{2} f_0 r_{1,2} r_1^{-1}, \tag{8A}$$

where $r_{1,2}$ is the derivative evaluated at O of the principal radius of normal curvature in the x_1-direction with respect to arc length on the line of curvature in the x_2-direction. These results yield Eq. (8) of the text.

References

1. J. J. Gilvarry, S. H. Browne, and I. K. Williams, *Theory of blind navigation by dynamical measurements* J. App Phys. **21**, 753–761 (1950).
2. S. A. Schelkunoff, *Solution of linear and slightly non-linear differential equations*, Q. App. Math. **3**, 348–355 (1945).

3. J. J. Gilvarry and S. H. Browne, *Application of Liouville's approximation to the blind navigation problem*, J. App. Phys. 21, 1195–1196 (1950).
4. L. I. Schiff, *Quantum mechanics*, McGraw-Hill Book Co., New York, 1949, p. 178.
5. L. Brillouin, *A practical method for solving Hill's equation*, Q. App. Math. **6**, 167–178 (1948).
6. L. Brillouin, *The B.W.K. approximation and Hill's equation, II*, Q. App. Math., **7**, 363–380 (1950).
7. J. Jeans, *The mathematical theory of electricity and magnetism*, fifth ed., Cambridge University Press, Cambridge, 1925, p. 167.
8. W. C. Graustein, *Differential geometry*, Macmillan Co., New York, 1947, pp. 36, 39, 115, 204.
9. T. M. MacRobert, *Spherical harmonics*, Dover Publications, New York, 1947, p. 133.
10. E. T. Whittaker, *A treatise on the analytical dynamics of particles and rigid bodies*, fourth ed., Dover Publications, New York, 1937, p. 99.
11. W. D. Lambert, *The shape and size of the earth*, Bulletin of the National Research Council, No. 78, 1931, 134–143.

SOME REMARKS ON A CLASS OF EIGENVALUE PROBLEMS WITH SPECIAL BOUNDARY CONDITIONS*

BY
G. W. MORGAN
Brown University

1. Introduction. The study of a variety of physical problems frequently leads one to an eigenvalue problem. One very common eigenvalue problem, the Sturm-Liouville problem, consists of finding non-identically vanishing solutions of the differential equation

$$L[y(x)] + \lambda wy \equiv (py')' - qy + \lambda wy = 0, \qquad x_0 \leq x \leq x_1 \tag{1}$$

subject to the homogeneous boundary conditions

$$\begin{aligned} a_0 y'(x_0) - b_0 y(x_0) &= 0, \\ a_1 y'(x_1) + b_1 y(x_1) &= 0. \end{aligned} \tag{2}$$

Here p, q, w are functions of x which are positive in the domain and possess the required number of derivatives, λ is an unknown constant parameter, and a_0, b_0 etc. are constants, such that $a_0 b_0$, $a_1 b_1 \geq 0$.

In this paper we consider a problem which is identical with the Sturm-Liouville problem in all respects except for the appearance of the parameter λ in the boundary conditions. This introduces certain difficulties. Our primary object is to discuss a transformation which eliminates λ from the boundary conditions and which thus enables us to carry over several important results of the Sturm-Liouville theory.

The method is then extended to higher order differential equations and finally physical examples are presented which provide an interpretation of the mathematical technique.

2. Comparison with the Sturm-Liouville Problem. The Sturm-Liouville problem is known to have the following properties.

Non-trivial solutions (eigenfunctions) exist only for certain positive values of λ (the eigenvalues). The eigenvalues λ_1, $\lambda_2 \cdots \lambda_i \cdots$, in order of magnitude, form a countably infinite set and the corresponding eigenfunctions y_1, $y_2 \cdots y_i \cdots$ constitute a complete, orthogonal system. Since the functions y_i are determined up to an arbitrary constant multiplier only, they can be normalized. The orthogonality and normalization conditions are then expressed by

$$\int_{x_0}^{x_1} wy_i y_j \, dx = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}. \tag{3}$$

By completeness we mean that any function $f(x)$ which is piecewise continuous can be approximated as closely as desired in the mean square error sense by a sufficiently large number of terms of the finite series $\sum_{i=1}^{n} c_i y_i$, where the coefficients c_i are given by

$$c_i = \int_{x_0}^{x_1} wfy_i \, dx. \tag{4}$$

*Received April 30, 1952.

The problem we wish to study here consists of the differential equation (1) together with the following boundary conditions:

$$y'(x_0) + K_0\lambda y(x_0) = 0,$$

$$y'(x_1) - K_1\lambda y(x_1) = 0,$$

$$\text{where} \qquad K_0 , K_1 \geq 0. \tag{5}$$

We first note that the usual orthogonality relation (3) no longer holds. In fact it can be shown following standard procedures that

$$\int_{x_0}^{x_1} wy_iy_j\,dx = -K_1p(x_1)y_i(x_1)y_j(x_1) - K_0p(x_0)y_i(x_0)y_j(x_0), \qquad i \neq j. \tag{6}$$

For the Sturm-Liouville problem we know that a function f which satisfies certain conditions* can be expanded in a series

$$f = \sum_{i=1}^{\infty} c_iy_i$$

where the expansion coefficients c_i are given by (4). In our problem this is no longer true.

Before discussing the transformation of the problem which will give, among other results, a physical interpretation of (6) and will also readily yield a formula for the expansion coefficients, we show how these coefficients can be derived formally by assuming the validity of the expansion

$$f(x) = \sum_{i=1}^{\infty} \gamma_iy_i , \qquad x_0 \leq x \leq x_1 . \tag{7}$$

Multiply by wy_j and integrate over the domain, reversing the order of summation and integration:

$$\int_{x_0}^{x_1} wfy_j\,dx = \sum_{i=1}^{\infty} \gamma_i \int_{x_0}^{x_1} wy_jy_i\,dx.$$

Using (6) we obtain

$$\int_{x_0}^{x_1} wfy_j\,dx = \gamma_j \int_{x_0}^{x_1} wy_j^2\,dx - \sum_{i=1}^{\infty}{}' \gamma_i[K_1p(x_1)y_i(x_1)y_j(x_1) + K_0p(x_0)y_i(x_0)y_j(x_0)]$$

where $\sum'$ indicates a summation in which the term $i = j$ is omitted. To eliminate the coefficients γ_i for $i \neq j$ we make use of the series (7) evaluated at the end points.

$$f(x_0) = \gamma_jy_j(x_0) + \sum_{i=1}^{\infty}{}' \gamma_iy_i(x_0),$$

$$f(x_1) = \gamma_jy_j(x_1) + \sum_{i=1}^{\infty}{}' \gamma_iy_i(x_1).$$

*See for example Courant Hilbert, Methoden der Math. Physik, Vol. I, Ch. VI, Sect. 3.3.

We now have

$$\int_{x_0}^{x_1} wfy_i\,dx = \gamma_i \int_{x_0}^{x_1} wy_i^2\,dx - K_0p(x_0)y_i(x_0)[f(x_0) - \gamma_i y_i(x_0)]$$
$$- K_1p(x_1)y_i(x_1)[f(x_1) - \gamma_i y_i(x_1)]. \tag{8}$$

If we require the y_i to be normalized so that

$$\int_{x_0}^{x_1} wy_i^2\,dx + K_0p(x_0)y_i^2(x_0) + K_1p(x_1)y_i^2(x_1) = 1, \tag{9}$$

then (8) gives

$$\gamma_i = \int_{x_0}^{x_1} wfy_i\,dx + K_0p(x_0)y_i(x_0)f(x_0) + K_1p(x_1)y_i(x_1)f(x_1). \tag{10}$$

The above is essentially a generalization of the method employed by some authors in the special case when the y_i are the trigonometric functions.*

3. Transformation to Quasi-Sturm-Liouville Form. We now show how the problem can be transformed into, essentially, the standard Sturm-Liouville form. We shall here use heuristic arguments only, although it is possible to put the reasoning on a more rigorous basis by means of suitable limit processes.

We define two functions $\delta(x_1 - \eta)$ and $\delta(\eta - x_0)$ in the following manner:

$$\delta(x_1 - \eta) = 0 \quad \text{for} \quad \eta \neq x_1\,,$$

and

$$\int_{x_0}^{x_1} v(\eta)\,\delta(x_1 - \eta)\,d\eta = v(x_1).$$

Similarly:

$$\delta(\eta - x_0) = 0 \quad \text{for} \quad \eta \neq x_0\,,$$

and

$$\int_{x_0}^{x_1} v(\eta)\,\delta(\eta - x_0)\,d\eta = v(x_0).$$

It is also convenient to define:

$$\begin{aligned} h(\xi) &\equiv \int_{x_0}^{\xi} y(\eta)\,\delta(x_1 - \eta)\,d\eta = \begin{cases} 0; \ x_0 \le \xi < x_1 \\ y(x_1); \quad \xi = x_1 \end{cases}, \\ g(\xi) &\equiv \int_{\xi}^{x_1} y(\eta)\,\delta(\eta - x_0)\,d\eta = \begin{cases} 0; \ x_0 < \xi \le x_1 \\ y(x_0); \quad \xi = x_0 \end{cases}. \end{aligned} \tag{11}$$

We now introduce a new dependent variable u by the relation:

$$u(x) = y(x) - \lambda K_1 \int_{x_0}^{x}\int_{x_0}^{\xi} y(\eta)\,\delta(x_1 - \eta)\,d\eta\,d\xi - \lambda K_0 \int_{x}^{x_1}\int_{\xi}^{x_1} y(\eta)\,\delta(\eta - x_0)\,d\eta\,d\xi. \tag{12}$$

---

*Timoshenko; Vibration Problems in Engineering, p. 319. K. Ludwig; Einwirkungen von Bodenschwingungen auf Pfeiler, Z. a. M. M. 14, 361-363 (1934).

From (11) we see that

$$\int_{x_0}^{x} h(\xi)\,d\xi = 0, \qquad x_0 \leq x \leq x_1\,,$$

$$\int_{x}^{x_1} g(\xi)\,d\xi = 0, \qquad x_0 \leq x \leq x_1\,. \tag{13}$$

Hence (12) becomes $u(x) = y(x)$, $x_0 \leq x \leq x_1$.

We now find the differential equation and boundary conditions satisfied by the function $u(x)$.

From equations (12) and (11) we have:

$$u'(x) = y'(x) - \lambda K_1 \int_{x_0}^{x} y(\eta)\,\delta(x_1 - \eta)\,d\eta + \lambda K_0 \int_{x}^{x_1} y(\eta)\,\delta(\eta - x_0)\,d\eta, \tag{14}$$

or

$$u'(x) = y'(x) - \lambda K_1 h(x) + \lambda K_0 g(x),$$

and therefore

$$u'(x) = y'(x), \qquad x_0 < x < x_1\,,$$

$$u'(x_1) = y'(x_1) - \lambda K_1 y(x_1), \tag{15}$$

$$u'(x_0) = y'(x_0) + \lambda K_0 y(x_0).$$

Comparing with (5), the boundary conditions on y, we see that our boundary conditions on u are

$$u'(x_0) = u'(x_1) = 0. \tag{16}$$

From (14) and (13) we obtain:

$$u''(x) = y''(x) - \lambda K_1\,\delta(x_1 - x)y(x) - \lambda K_0\,\delta(x - x_0)y(x)$$

$$= y''(x) - \lambda K_1\,\delta(x_1 - x)u(x) - \lambda K_0\,\delta(x - x_0)u(x).$$

Substitution into the differential equation (1) yields

$$(pu')' - qu + \lambda[wu + K_0 p\,\delta(x - x_0)u(x) + K_1 p\,\delta(x_1 - x)u(x)$$

$$- K_0 p' g(x) + K_1 p' h(x)] = 0.$$

Now g and h vanish everywhere except at the end points, and at the end points they are irrelevant in view of the presence of the terms containing $\delta(x - x_0)$ and $\delta(x_1 - x)$, respectively. Hence the differential equation for u can be written:

$$(pu')' - qu + \lambda\rho u = 0 \tag{17}$$

where $\rho = w + K_0 p(x_0)\,\delta(x - x_0) + K_1 p(x_1)\,\delta(x_1 - x)$.

The problem defined by (16) and (17) is in the standard Sturm-Liouville form except for the singularities of the function ρ at the ends. We shall formally regard it as a Sturm-Liouville problem and assume without attempting a rigorous justification that many results of the Sturm-Liouville theory, such as those concerning the nature of the eigenvalues, the completeness of the eigenfunctions, etc. are still applicable.

The orthogonality (3) yields

$$\int_{x_0}^{x_1} \rho u_i u_j \, dx = \int_{x_0}^{x_1} \rho y_i y_j \, dx = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}.$$

Substituting for $\rho(x)$, the integral becomes

$$\int_{x_0}^{x_1} w y_i y_j \, dx + K_0 p(x_0) \int_{x_0}^{x_1} y_i y_j \, \delta(x - x_0) \, dx + K_1 p(x_1) \int_{x_0}^{x_1} y_i y_j \, \delta(x_1 - x) \, dx,$$

and this gives

$$\int_{x_0}^{x_1} w y_i y_j \, dx + K_0 p(x_0) y_i(x_0) y_j(x_0) + K_1 p(x_1) y_i(x_1) y_j(x_1) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}. \tag{18}$$

This is immediately seen to be identical with (6) and the previously introduced normalization (9).

The coefficients γ_i in the approximation

$$f \sim \sum_{i=1}^{n} \gamma_i u_i = \sum_{i=1}^{n} \gamma_i y_i$$

are immediately found by using the appropriate functions in formula (4);

$$\begin{aligned} \gamma_n &= \int_{x_0}^{x_1} \rho f u_n \, dx = \int_{x_0}^{x_1} \rho f y_n \, dx \\ &= \int_{x_0}^{x_1} w f y_n \, dx + K_0 p(x_0) f(x_0) y_n(x_0) + K_1 p(x_1) f(x_1) y_n(x_1), \end{aligned}$$

and this is identical with (10).

Completely analogous results can of course be derived if λ only appears in one of the boundary conditions, the other condition being as in (2). In that case only one of the two double integrals in (12) will have to be used in the definition of the new variable $u(x)$.

4. Extension to Higher Order Self-Adjoint Differential Equations. An analogous method applies to the equation

$$(p_2 y'')'' - (p_1 y')' + pu - \lambda w u = 0, \qquad x_0 \leq x \leq x_1 . \tag{91}$$

This is to be solved subject to four homogeneous boundary conditions, one or two of which involve the parameter λ in the following manner:

$$\begin{gathered} y'''(x_0) - K_0 \lambda y(x_0) = 0, \\ y'''(x_1) + K_1 \lambda y(x_1) = 0, \\ K_0 , K_1 \geq 0. \end{gathered} \tag{20}$$

As an illustration consider the case in which, in addition to (20), the conditions $y''(x_0) = y''(x_1) = 0$ are to be satisfied.

Define the new variable $u(x)$ by

$$\begin{aligned} u(x) = y(x) &+ \lambda K_1 \int_{x_0}^{x} \int_{x_0}^{r} \int_{x_0}^{s} \int_{x_0}^{\xi} y(\eta) \, \delta(x_1 - \eta) \, d\eta \, d\xi \, ds \, dr \\ &+ \lambda K_0 \int_{x}^{x_1} \int_{r}^{x_1} \int_{s}^{x_1} \int_{\xi}^{x_1} y(\eta) \, \delta(\eta - x_0) \, d\eta \, d\xi \, ds \, dr. \end{aligned} \tag{21}$$

Proceeding as before we obtain

$$u(x) = y(x),$$
$$u'(x) = y'(x), \tag{22}$$
$$u''(x) = y''(x), \qquad x_0 \le x \le x_1 .$$

Also

$$u'''(x) = y'''(x) + \lambda K_1 h(x) - \lambda K_0 g(x)$$

where g and h are defined as before by (11). Hence:

$$u'''(x) = y'''(x), \qquad x_0 < x < x_1 ,$$
$$u'''(x_1) = y'''(x_1) + \lambda K_1 y(x_1), \tag{23}$$
$$u'''(x_0) = y'''(x_0) - \lambda K_0 y(x_0).$$

Comparing (22) and (23) with (20) we see that the boundary conditions on u are:

$$u''(x_0) = u'''(x_0) = u''(x_1) = u'''(x_1) = 0. \tag{24}$$

Finally we have:

$$u^{\mathrm{IV}} = y^{\mathrm{IV}} + \lambda K_1 \,\delta(x_1 - x)y(x) + \lambda K_0 \,\delta(x - x_0)y(x).$$

Substituting in the differential equation (19), we obtain

$$(p_2 u'')'' - (p_1 u')' + pu - \lambda \rho u = 0, \tag{25}$$

where $\rho = w + K_0 p_2(x_0)\,\delta(x - x_0) + K_1 p_2(x_1)\,\delta(x_1 - x)$.

Hence we have again succeeded in converting the original problem into a standard form, the differential equation having been changed only by the replacement of the original weight function w by a new weight function ρ identical with w everywhere except for singularities at the boundaries. We are again in a position to apply the results of the standard theory.

The orthogonality relations and the formulas for the coefficients γ_i in the approximation

$$f \sim \sum_{i=1}^{n} \gamma_i u_i = \sum_{i=1}^{n} \gamma_i y_i$$

are found precisely as before.

This technique can be extended in a self-evident manner to self-adjoint differential equations of order $2m$, $m = 1, 2, \cdots$ if one or two of the boundary conditions are of the form

$$y^{(2m-1)}(x_i) + K_i \lambda y(x_i) = 0, \qquad i = 0, 1$$

and the K_i have the proper sign to make $\rho(x)$ positive.

5. Physical Interpretation. We now consider three illustrative examples of physical systems, the mathematical analysis of which leads to the type of eigenvalue problem under discussion.

(a) Longitudinal vibrations of a bar with a mass at the end.

The differential equation is

$$w \frac{\partial^2 v}{\partial t^2} = \frac{\partial}{\partial x}\left(AE \frac{\partial v}{\partial x}\right) \tag{26}$$

where v is the longitudinal displacement of a cross section of the bar, w is the mass per unit length, A is the cross section, and E is Young's modulus.

If the bar is built in at $x = 0$, the boundary condition there is $v(0,t) = 0$. If a mass M is attached to the end $x = l$, the force in the bar at that end must be equal to the rate of change of momentum of the mass M.

Hence the boundary condition at $x = l$ is:

$$AE \frac{\partial v}{\partial x} = -M \frac{\partial^2 v}{\partial t^2} \quad \text{at} \quad x = l, \quad t > 0. \tag{27}$$

To solve this problem we look for normal modes in the form

$$v(x, t) = \left.\begin{matrix} \sin \omega t \\ \cos \omega t \end{matrix}\right\} \cdot y(x)$$

and hence obtain the following differential equation and boundary conditions for y:

$$(AEy')' + \lambda wy = 0,$$

$$y = 0 \quad \text{at} \quad x = 0,$$

$$AEy' - \lambda My = 0 \quad \text{at} \quad x = l$$

$$\text{where} \quad \lambda = \omega^2. \tag{28}$$

This is precisely of the form (1) and (5) with a standard boundary condition at $x = 0$.

Transformation of this problem to the form (16) and (17) gives:

$$(AEu')' + \lambda(w + M\ \delta(l - x))u = 0,$$

$$u = 0 \quad \text{at} \quad x = 0,$$

$$u' = 0 \quad \text{at} \quad x = l. \tag{29}$$

The physical meaning of the formulation (29) is that the mass M is treated as a part of the elastic system by making the mass per unit length infinite at $x = l$ in such a manner that the total mass added is M. The new boundary condition at $x = l$ is now applied at the outer end of the mass M and is the condition for a free end.

The new orthogonality condition, given by (18) is:

$$\int_0^l [(w + M\ \delta(l - x)]y_i y_j\, dx = \int_0^l wy_i y_j\, dx + My_i(l)y_j(l) = 0, \qquad i \neq j. \tag{30}$$

It clearly shows the physical meaning of the transformation. The infinite mass density at $x = l$ changes the weight function to account for the extra mass M.

(b) Lateral vibrations of a beam with end load.*

*F. Gassmann, Über Querschwingungen eines Stabes mit Einzelmasse, Ingenieur—Archiv 2, 222-227, 1931.

The mathematical formulation of this problem is:

$$w \frac{\partial^2 v}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 v}{\partial x^2}\right) = 0 \tag{31}$$

where v is the lateral displacement of a cross-section and EI is the flexural rigidity.

If the beam carries a mass M at $x = l$, the boundary conditions there are:

$$EI \frac{\partial^3 v}{\partial x^3} = M \frac{\partial^2 y}{\partial t^2},$$

and

$$\frac{\partial^2 v}{\partial x^2} = 0. \tag{32}$$

These conditions assume that the center of gravity of the mass M is very close to $x = l$ so that terms arising from this distance can be neglected.

The search for normal modes leads to:

$$(EIy'')'' - \lambda wy = 0, \tag{33}$$

and the boundary conditions at $x = l$ become:

$$\begin{aligned} EIy''' + \lambda My &= 0, \\ y'' &= 0. \end{aligned} \tag{34}$$

This is of the form discussed in section 4. If we carry out the transformation we again see that its physical interpretation is the inclusion of the mass M into the elastic system with the effect of obtaining a new weight function $w + M\,\delta(l - x)$.

(c) Heat Conduction in a rod with finite energy reservoir at end.

Consider one end of the rod to be in contact with a body in which heat diffusion is so rapid that it is permissible to consider the body to be at a uniform temperature equal to that at the point of contact. We also assume that the only energy flux into, or out of, the body occurs at the contact with the rod.

We have

$$cm \frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(KA \frac{\partial v}{\partial x}\right) \tag{35}$$

where v is the temperature, K the termal conductivity, c the specific heat per unit mass, m the mass of the rod per unit length. If, at $x = l$, the rod is in contact with a body of mass M and specific heat γ, then the boundary condition there is:

$$-KA \frac{\partial v}{\partial x} = \gamma M \frac{\partial v}{\partial t}, \qquad x = l. \tag{36}$$

We try solutions:

$$v = e^{-\lambda t} y(x).$$

The equation for y is:

$$(KAy')' + \lambda(cm)y = 0. \tag{37}$$

The boundary condition at $x = l$ is:

$$KAy' - \lambda\gamma My = 0. \tag{38}$$

Comparing with (1) and (5) we see that

$$KA = p, \qquad cm = w, \qquad \text{and} \qquad \gamma M/KA = K_1 . \tag{39}$$

Our transformation yields:

$$(KAu')' + \lambda(cm + \gamma M\ \delta(l - x))u = 0,$$

and

$$u' = 0 \qquad \text{at} \qquad x = l. \tag{40}$$

The physical interpretation is the same as in the previous examples. The body at the end is thought of as a part of the system in which conduction takes place. This is done by making the heat capacity per unit length infinite at $x = l$ in such a manner that a total heat capacity γM is added to the system at that end.

THE CHEBYSHEV APPROXIMATION METHOD*

BY

PRESTON R. CLEMENT**

Electrical Engineering Department, Princeton University

A treatment of the general Chebyshev approximation method as it interests physicists and engineers is given, with a detailed discussion of the properties of Chebyshev polynomials. Brief applications to electric circuit theory are presented.***

Introduction. Of the various means of approximating a given function, the Chebyshev method is, for physicists and engineers, one of the most interesting and important. Originally introduced by P. L. Chebyshev in a mechanical linkage problem, (26) this procedure came into particular importance in electrical engineering with the publication of a new method of filter design by W. Cauer.(5)

It is the aim of this paper to review the various articles written on this subject, with the purpose of presenting the material in a manner sufficiently general to permit physicists and engineers to appreciate fully this approximation problem. Particular emphasis is given to Chebyshev polynomials, with brief applications to electric circuit theory.

1. The Problem of Approximation. Chebyshev Approximation. Consider a function $f(x)$ defined in an interval $a \leqq x \leqq b$. Let $g_n(x; a_i)$ be a function defined in the same interval with n adjustable parameters $a_1, \cdots, a_n$. Our problem is the following: What should be the values of the n parameters in order that $g_n(x; a_i)$ will approximate $f(x)$ as well as possible in the interval (a, b)?

The answer to this question depends, of course, on what is meant by the expression "as well as possible." The applied mathematician is interested in obtaining a function $g_n(x; a_i)$ which will be easier to use in his calculations than the original function $f(x)$, yet which will produce a result sufficiently close to the exact answer. A frequently used approximation is that of least squares; that is, the a's are adjusted so that

$$\int_a^b [f(x) - g_n(x; a_i)]^2 \, dx$$

is a minimum. For example, if the interval (a, b) is $(-\pi, \pi)$, if $f(x)$ is odd about the origin, and if $g_n(x; a_i) = a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx$, then the least squares approximation will produce the Fourier coefficients for the a's.

Let us now consider an engineering problem, the design of filter networks on an amplitude spectrum basis. Suppose that it is desired to have a transmission characteristic $f(x)$ which has a constant non-zero value in the pass bands and a zero value everywhere else. It is clear that with a physically realizable circuit such a characteristic can only be approximated. Thus in this case, we can denote by $g_n(x; a_i)$ the amplitude spectrum of the actual filter circuit which should approximate our ideal characteristic $f(x)$ "as well as possible." x will denote the frequency, and the a_i will be numbers determined by the adjustable circuit parameters of the system.

For the purpose of filtering, it is clear that the least squares approximation may not

*Received July 18, 1952.

**Now on military leave.

***This research was supported in part by funds of the Eugene Higgins Trust allocated to Princeton University.

be a very satisfactory one, since one may desire the guarantee that in the stopbands the attenuation for *every* frequency is above a certain definite value. With the least squares approximation, the attenuation in the mean of a frequency mixture in the stopbands is indeed very high, but for one or more frequencies it may be quite small.

Thus to insure that such large departures from the ideal characteristic do not occur for *any* frequency, we would demand that the maximum of

$$| f(x) - g_n(x; a_i) |$$

be a minimum in the prescribed interval (a, b). When such is the case, $g_n(x; a_i)$ is said to approximate $f(x)$ *in the Chebyshev sense* in the interval (a, b). Hereafter, we will mean by "approximate," approximate in the Chebyshev sense.

If we let $h_n(x; a_i) = f(x) - g_n(x; a_i)$, our general problem becomes the following:

Given an arbitrary function $h_n(x; a_i)$ with n parameters $a_1, \cdots, a_n$, what should be the values of these parameters to make the maximum difference from zero of $h_n(x; a_i)$ a minimum for $a \leqq x \leqq b$; i.e., *so that $h_n(x; a_i)$ approximates zero in (a, b)?*

2. The Function $h_n(x;a_i)$. In all that follows in this section, it will be assumed that $h_n(x; a_i)$ is an arbitrary function defined in the interval $a \leqq x \leqq b$, subject to the restriction that the function and its derivatives with respect to $x, a_1, \cdots, a_n$ are finite and continuous. In order to determine the values of the parameters for the maximum deviation of $h_n(x; a_i)$ from zero to be a minimum, the actual value of this deviation, and the points at which this deviation is assumed, two sets of equations will be considered.

The first set arises simply by noting that if L_n is the smallest possible maximum deviation of $h_n(x; a_i)$ from zero in (a, b), then when the parameters are *properly adjusted*, the equation

$$h_n^2(x;a_i) - L_n^2 = 0$$

will give the set of points $x_1, x_2, \cdots, x_\mu$ in (a, b) at which the maximum is assumed. If the value L_n is assumed on the boundary of the interval, then two of the points of the set $x_1, \cdots, x_\mu$ are a and b. The rest of the points are on the interior, where the first derivative must vanish.Hence if differentiation with respect to x is denoted by a prime, the first set of equations to be satisfied is

$$\left.\begin{aligned} h_n^2(x; a_i) - L_n^2 &= 0 \\ (x - a)(x - b)h_n'(x; a_i) &= 0. \end{aligned}\right\} \qquad (1)$$

These equations will have μ common solutions which are real and distinct; $x_1, x_2, \cdots, x_\mu$.

The second set of equations comes from a theorem stating a necessary condition on $h_n(x; a_i)$ in order for the function to satisfy the Chebyshev condition.

Theorem 1. Given a function $h_n(x; a_i)$ continuous along with its derivatives in the interval $a \leqq x \leqq b$. Let $x_1, \cdots, x_\mu$ denote the set of points in (a, b) at which $| h_n(x; a_i) |$ assumes a maximum. Let the greatest of these maxima be denoted by M. Then if $M = M(a_1, a_2, \cdots, a_n)$ has a minimum value for the parameters specified (i.e., $M = L_n$), the following set of equations is *insoluble* for every set of non-zero real numbers $S_1, S_2, \cdots, S_\mu$ such that the sign of S_k is the same as that of $h_n(x_k; a_i)$, $k = 1, 2, \cdots, \mu$.

$$\left.\begin{array}{l}
\dfrac{\partial h_n(x_1\ ;\ a_i)}{\partial a_1}\lambda_1 + \dfrac{\partial h_n(x_1\ ;\ a_i)}{\partial a_2}\lambda_2 + \cdots + \dfrac{\partial h_n(x_1\ ;\ a_i)}{\partial a_n}\lambda_n = S_1 \\[2ex]
\dfrac{\partial h_n(x_2\ ;\ a_i)}{\partial a_1}\lambda_1 + \dfrac{\partial h_n(x_2\ ;\ a_i)}{\partial a_2}\lambda_2 + \cdots + \dfrac{\partial h_n(x_2\ ;\ a_i)}{\partial a_n}\lambda_n = S_2 \\[2ex]
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\[2ex]
\dfrac{\partial h_n(x_\mu\ ;\ a_i)}{\partial a_1}\lambda_1 + \dfrac{\partial h_n(x_\mu\ ;\ a_i)}{\partial a_2}\lambda_2 + \cdots + \dfrac{\partial h_n(x_\mu\ ;\ a_i)}{\partial a_n}\lambda_n = S_\mu
\end{array}\right\} \tag{2}$$

A theorem similar to this was proved by Chebyshev (27). The extension to the present form was given by Kircherberger (20).

Example 1. $h_2(x; a_i) = x^2 + a_1x + a_2$, $(-1 \leqq x \leqq 1)$.

For this function, equations (1) become:

$$(x^2 + a_1x + a_2)^2 - L_2^2 = 0, \tag{3}$$

$$(x - 1)(x + 1)(2x + a_1) = 0. \tag{4}$$

Substituting two solutions of (4), $x = +1$ and $x = -1$, into (3) and subtracting the two resulting equations:

$$a_1 + a_1a_2 = 0. \tag{5}$$

Taking a_2 equal to -1, which certainly satisfies (5), the desired polynomial appears to be

$$x^2 \pm 2x - 1 \tag{6}$$

with a maximum deviation in the interval $(-1, 1)$ of 2.

Let us use the plus sign in (6). Then the polynomial has a maximum at $+1$ of $+2$ and a minimum at -1 of -2. Equations (2) then become

$$-\lambda_1 + \lambda_2 = -S_1$$

$$\lambda_1 + \lambda_2 = S_2$$

where S_1, S_2 are positive. Thus

$$\lambda_1 = (S_2 + S_1)/2,$$

$$\lambda_2 = (S_2 - S_1)/2,$$

violating the necessary condition for Chebyshev approximation given by theorem 1 that equations (2) be insoluble. The same would apply if the negative sign had been chosen in (6). Thus (6) is not the desired polynomial.

To solve our problem we note that another solution of (5) is possible, namely, $a_1 = 0$. Substitution of this value for a_1 gives for a_2 the value of $-\frac{1}{2}$. Hence the polynomial becomes

$$x^2 - \tfrac{1}{2} \tag{7}$$

with a maximum deviation of $\frac{1}{2}$ at $x = \pm 1, 0$. For this function equations (2) become

$$
\begin{aligned}
-\lambda_1 + \lambda_2 &= S_1 , \\
\lambda_2 &= -S_2 , \\
\lambda_1 + \lambda_2 &= S_3 ,
\end{aligned}
\tag{8}
$$

where S_1, S_2, S_3 are greater than zero. It is clear that (8) is insoluble in the λ's. Thus the necessary condition that the polynomial approximate zero in the Chebyshev sense is satisfied. That this is the proper polynomial of second degree follows from the fact that the maximum deviations are equal and three in number. This property of polynomials will be shown in section III.

Example 2. $h_1(x; a_1) = x + a_1 \sin (7x/2), (-\pi \leqq x \leqq \pi)$.

Again the problem is to find the value of a_1 such that the maximum difference of this function from zero in the interval $(-\pi, \pi)$ is a minimum.

Equations (1) becomes

$$x^2 + 2a_1 x \sin (7x/2) + a_1^2 \sin^2 (7x/2) - L_1^2 = 0, \tag{9}$$

$$(x - \pi)(x + \pi)[1 + (7/2)a_1 \cos (7x/2)] = 0. \tag{10}$$

By a process similar to that used in example 1, it can be shown that the function $x + a_1 \sin (7x/2)$ which satisfies the Chebyshev condition in the interval $(-\pi, \pi)$ is

$$h_1(x; a_1) = x + 0.39 \sin (7x/2),$$

with its maximum deviation equal to 2.75. Its graph is shown in Fig. 1.

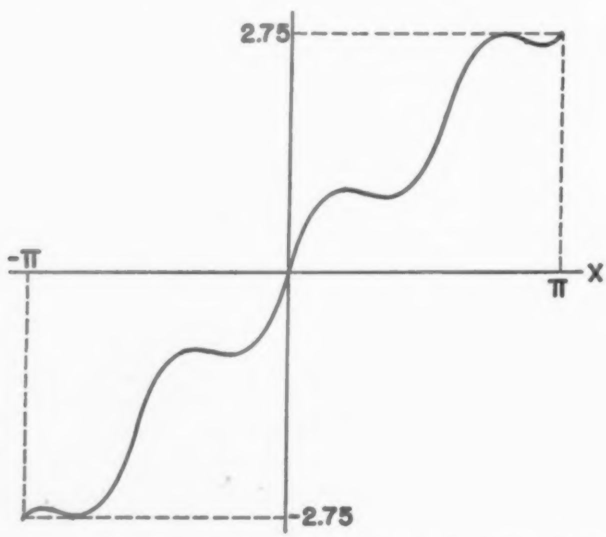

FIG. 1. The function $x + a_1 \sin (7x/2)$ which satisfies the Chebyshev condition in the interval $(-\pi, \pi)$.

Example 2 was presented primarily as an illustration of a function which satisfies the Chebyshev condition yet whose relative maxima are not all equal. For many general functions, however, the Chebyshev condition is satisfied only when all the relative maxima are equal. This statement is made more precise in the following theorem.

Theorem 2.* Given a function $h_n(x; a_i)$ defined and continuous along with all its derivatives in the interval (a, b). Let μ be the number of ordinates of $h_n(x; a_i)$ where the magnitude of the function is a relative maximum, including the end points a and b and let $\mu = n + 1$, where n is the number of parameters. It is further assumed that either the first derivatives of $h_n(x; a_i)$ with respect to the a_i do not vanish at $x_1, \cdots, x_\mu$ or else if the first derivative vanishes, the first non-vanishing derivative is of odd order. Then if $h_n(x; a_i)$ satisfies the Chebyshev condition in (a, b) the μ relative maxima are all equal.

This so-called equal maximum property of Chebyshev functions will also occur in certain special cases where $\mu \neq n + 1$. For instance, in example 2 if the antisymmetrical function had been taken as $x + a_1 \sin(5x/2)$ in the interval $(-\pi, \pi)$, then all four maximum deviations would be equal under the Chebyshev condition, even though there is only one parameter to adjust.

We will now leave the general Chebyshev function and consider the important special case of the Chebyshev polynomials.

3. Chebyshev Polynomials. Given a function $f(x)$ which is finite and continuous along with all its derivatives in the interval (a, b). Let $g_n(x; a_i) = a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$ be the approximating function. Hence the function which we desire to approximate zero in the Chebyshev sense is

$$h_n(x; a_i) = f(x) - [a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n]. \tag{11}$$

Let $x_1, x_2, \cdots, x_\mu$ be the set of points in the interval (a, b) at which the function given by (11) assumes its maximum deviation L_n. Under these conditions, equations (2) become

$$\left.\begin{array}{l} \lambda_1 x_1^{n-1} + \lambda_2 x_1^{n-2} + \cdots + \lambda_{n-1}x_1 + \lambda_n = S_1 \\ \lambda_1 x_2^{n-1} + \lambda_2 x_2^{n-2} + \cdots + \lambda_{n-1}x_2 + \lambda_n = S_2 \\ \cdot \\ \cdot \\ \cdot \\ \lambda_1 x_\mu^{n-1} + \lambda_2 x_\mu^{n-2} + \cdots + \lambda_{n-1}x_\mu + \lambda_n = S_\mu \end{array}\right\}. \tag{12}$$

Now let $P(x)$ denote $\lambda_1x^{n-1} + \lambda_2x^{n-2} + \cdots + \lambda_n$, an entire rational function of $(n - 1)$th degree, which by system (12) is determined only in so far as that at a prescribed point it has a prescribed sign. To settle the question of whether equations (12) can be satisfied by a suitable choice of coefficients, it is necessary to consider the number of changes of sign of $S_1, S_2, \cdots, S_\mu$. If there is a change of sign in going from S_k to S_{k+1}, then there is a root of $P(x) = 0$ between x_k and x_{k+1}. Since we have $(n - 1)$ roots at our disposal, and since the number of changes in sign can never exceed the number of roots, then it follows that if the number of sign changes in the S's is less than or equal to $(n - 1)$,

*The statement of this theorem and its proof follows along the lines of one given by LeCorbeiller (22), except that it was found necessary to add a hypothesis to take care of special functions.

system (12) will be soluble for the λ's. Hence the following proposition has been demonstrated;

A necessary condition for the function $h_n(x; a_i)$ as given by (11) to approximate zero in the Chebyshev sense is that among the set of values $h_n(x_1; a_i)$, $h_n(x_2; a_i)$, $\cdots$, $h_n(x_\mu; a_i)$ there are at least n changes in sign.

From this it follows that if $h_n(x; a_i)$ as given by (11) is to approximate zero, then equation set (1) will have at least $(n + 1)$ common solutions in (a, b), and further the maximum deviation L_n will be alternately a maximum and a minimum of the function.

To show that the condition of the non-solubility of equations (2) is not only necessary but also sufficient for the function given by (11), it will be shown that the condition leads to a unique function $h_n(x; a_i)$.

To prove this, assume that there exists a function $\bar{h}_n(x; a_i) = f(x) - (\bar{a}_1 x^{n-1} + \bar{a}_2 x^{n-2} + \cdots + \bar{a}_n)$ which also satisfies the non-solubility condition, in other words, whose maximum deviation is less or equal L_n. Thus it follows that $(h_n - \bar{h}_n)$ is a polynomial of degree at most $(n - 1)$. By the previous discussion there are at least $(n + 1)$ points at which $|h_n| = L_n$ and at least $(n + 1)$ points at which $|\bar{h}_n| \leqq L_n$. Thus $(h_n - \bar{h}_n)$ has at least n zero points in (a, b). This means that necessarily $h_n \equiv \bar{h}_n$.

Let us now take for $f(x)$ the simplest possible form:

$$f(x) = x^n. \tag{13}$$

Thus the function which we wish to approximate zero may be written as

$$h_n(x; b_i) = x^n + b_1 x^{n-1} + b_2 x^{n-2} + \cdots + b_n \tag{14}$$

where the b's of equation (14) are simply the negatives of the corresponding a's of equation (11).

For convenience, let us repeat equations (1).

$$h_n^2(x; b_i) - L_n^2 = 0, \tag{1a}$$

$$(x - a)(x - b)h_n'(x; b_i) = 0. \tag{1b}$$

Since equation (1b) is of degree $(n + 1)$, it follows from the previous discussion that all its roots must be distinct, and furthermore they must all also be roots of equation (1a). Now since the roots of $h_n'(x; b_i) = 0$ are roots of (1a), it follows that there are $(n - 1)$ double roots of (1a). Hence the system of equations

$$h_n^2(x; b_i) - L_n^2 = 0$$

$$[h_n'(x; b_i)]^2 = 0$$

will have $(2n - 2)$ common solutions in the form of pairs of double roots. The other two common solutions to (1a) and (1b) *must* occur at $x = a$ and $x = b$. Thus the equations

$$h_n^2(x; b_i) - L_n^2 = 0$$

$$(x - a)(x - b)[h_n'(x; b_i)]^2 = 0$$

will have $2n$ common solutions. When it is recognized that the coefficient of x^{2n} in the first of these equations is unity, while in the second it is n^2, it is seen that the differential

equation to be solved is

$$h_n^2 - L_n^2 = \frac{(x-a)(x-b)}{n^2}\left(\frac{dh_n}{dx}\right)^2. \tag{15}$$

There are four solutions to equation (15) which result from the four possible combinations of signs appearing after the square root of both sides has been taken. These solutions are

$$h_n = L_n \begin{matrix}\sin\\ \cos\end{matrix}\left\{n \text{ arc} \begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\left(\frac{-2x+a+b}{a-b}\right) + C\right\} \tag{16}$$

where C is the constant of integration. Since our problem is to determine the *polynomial* of nth degree which approximates zero best, we are restricted to the solution

$$h_n = L_n \cos\left[n \text{ arc} \cos\left(\frac{-2x+a+b}{a-b}\right) + C\right]. \tag{17}$$

The constant of integration is determined by the requirement that at the boundaries of the interval (a, b), the polynomial must assume its maximum deviation. This requirement is satisfied for $C = 0$, though, of course, placing C equal to an integral multiple of π would not change the function h_n.

In order to get (17) in a form which shows its polynomial character more readily, let

$$z = \text{arc} \cos\left(\frac{-2x+a+b}{a-b}\right).$$

Then

$$\begin{aligned} h_n &= L_n \cos nz = L_n/2[(\cos z + i \sin z)^n + (\cos z - i \sin z)^n] \\ &= \frac{L_n}{2(a-b)^n}[(-2x+a+b+\sqrt{4x^2-2(a+b)x+4ab})^n \\ &\qquad + (-2x+a+b-\sqrt{4x^2-2(a+b)x+4ab})^n]. \end{aligned} \tag{18}$$

For the coefficient of x^n to be unity, as given in (14),

$$L_n = \pm\frac{(a-b)^n}{2^{2n-1}}. \tag{19}$$

Tabulation of Chebyshev polynomials in the interval (a, b).

From equation (18) the following set of polynomials are obtained;

$$h_1 = x - \frac{a+b}{2}$$

$$h_2 = x^2 - \frac{3(a+b)}{4}x + \frac{a^2+6ab+b^2}{8}$$

$$h_3 = x^3 - \frac{9(a+b)}{8}x^2 + \frac{3a^2+12ab+3b^2}{8}x - \frac{a+b}{32}(a^2+14ab+b^2)$$

.
.
.

Chebyshev polynomials in the interval $(-1, 1)$.

It is usually convenient to deal with the Chebyshev polynomials in the interval $(-1, 1)$ instead of the interval (a, b). From (17) and (19), these functions may compactly be written as

$$\frac{1}{2^{n-1}} \cos (n \text{ arc} \cos x).$$

For our purposes the polynomials of unit maximum amplitude in the interval $(-1, 1)$ will be more useful: i.e., the polynomials given by

$$T_n(x) = \cos (n \text{ arc} \cos x). \tag{20}$$

Hereafter, references to Chebyshev polynomials will means those functions given by (20). The first ten such functions are tabulated in Table I for convenience. Fig. 2 shows a few of these polynomials.

TABLE I. TABULATION OF THE CHEBYSHEV POLYNOMIALS

$T_n(x) = \cos (n \text{ arc} \cos x)$

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$
$$T_4(x) = 8x^4 - 8x^2 + 1$$
$$T_5(x) = 16x^5 - 20x^3 + 5x$$
$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$
$$T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$$
$$T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$$
$$T_9(x) = 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$$
$$T_{10}(x) = 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1$$

Another convenient form for $T_n(x)$ is, from (18),

$$T_n(x) = \frac{1}{2} [(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n]. \tag{21}$$

Also, the expansion of (20) gives:

$$T_n(x) = \sum_{\lambda=0}^{[n/2]} (-1)^\lambda \binom{n-\lambda}{\lambda} \frac{n}{n-\lambda} \frac{x^{n-2\lambda}}{2^{2\lambda-n+1}} \qquad (n \geqq 1) \tag{22}$$

where $[n/2]$ denotes the largest integer contained in the set $0, \cdots, n/2$.

A still further form for $T_n(x)$, which may be checked by using the original differential equation, is

$$T_n(x) = 2^n(-1)^n \frac{n!}{(2n)!} \sqrt{1 - x^2} \frac{d^n}{dx^n} (1 - x^2)^{n-1/2} \qquad (n \geqq 0) \tag{23}$$

Further properties of the Chebyshev polynomials. 1. Recurrence relation. Letting again $z = \text{arc cos } x$, $T_n = \cos nz$. Since $\cos (n + 1)z + \cos (n - 1)z = 2 \cos nz \cos z$, then $T_{n+1} + T_{n-1} = 2T_nT_1$. Thus taking into account from (20) the values of T_n for $n = 0, 1$:

$$T_{n+1} = 2xT_n - T_{n-1} \tag{24}$$

where

$$T_0 = 1 \quad \text{and} \quad T_1 = x.$$

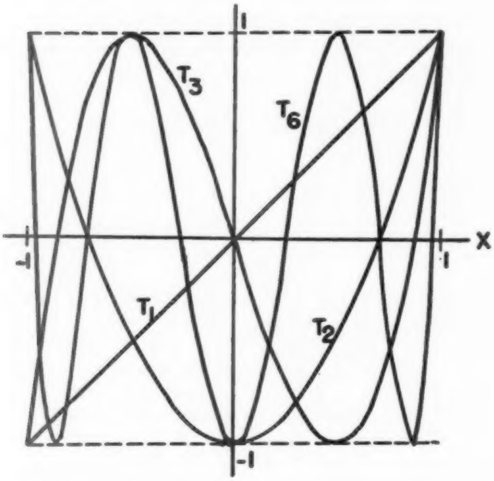

FIG. 2. The Chebyshev polynomials $T_n(x) = \cos (n \text{ arc cos } x)$.

2.Orthogonal property. The T's are othogonal to each other in the interval $(-1, 1)$ with respect to the function $1/\sqrt{1 - x^2}$, as is easily seen from the fact that

$$\int_{-1}^{1} T_n(x)T_m(x) \frac{dx}{\sqrt{1 - x^2}} = \int_0^{\pi} \cos n\theta \cos m\theta \, d\theta = \begin{cases} 0 \ (m \neq n) \\ \frac{\pi}{2} \ (m = n \geqq 1) \\ \pi \ (m = n = 0) \end{cases} \tag{25}$$

3. Generating functions. One generating function which may be used to define the $T_n(x)$ is:

$$\frac{1 - xt}{1 - 2xt + t^2} = \sum_{n=0}^{\infty} T_n(x)t^n. \qquad (t < 1)$$

A second such function is

$$\ln \frac{1}{\sqrt{t^2 - 2xt + 1}} = \sum_{n=1}^{\infty} T_n(x) \frac{t^n}{n}. \qquad (t < 1)$$

4. Maximization of the largest root. This property will be stated in the form of a

z direction, as shown in Fig. 4. Thus we may say that the curve is given in space by $z = \cos n\theta$ and $r = 1$. Therefore, the abscissa of the orthogonal projection of this curve on the plane formed by the z-axis and the $\theta = 0$ line is then $x = \cos \theta$. Hence the projected

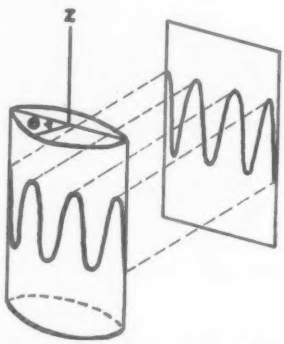

FIG. 4. The curve $\cos 7\theta$ on a cylinder of unit radius and its projection on a plane to form a Lissajous pattern.

figure gives the relation between $x = \cos \theta$ and $z = \cos n\theta$ and is thus represented by

$$z = \cos (n \text{ arc } \cos x).$$

If time is introduced as the variable instead of θ, the projection of the curve $z = \cos n\theta$ can be written in the parametric form

$$z = \cos nt,$$

$$x = \cos t,$$

and thus represent Lissajous figures for an integral ratio of the two composing frequencies.

If the curve had been taken as $z = \sin n\theta$, then the projections would have been proportional to the $U_n(x)$ functions instead of to the $T_n(x)$.

9. $T_n(x)$ and $U_n(x)$ in relation to Fourier series. A similar geometric interpretation can be given (30) to show the relation of the T_n and U_n functions to Fourier series. The connection can be seen, however, quite easily in the following way. Let $Y(\theta)$ be a given function in an interval, say (a, b). Developing this function in a Fourier series, we have

$$Y(\theta) = \sum_{n=0}^{\infty} (a_n \cos n\theta + b_n \sin n\theta). \qquad (a \leqq \theta \leqq b)$$

If x is set equal to $\cos \theta$:

$$Y(\cos^{-1} x) = \sum_{n=0}^{\infty} [a_n T_n(x) + b_n U_n(x)]. \qquad (a \leqq \cos^{-1} x \leqq b)$$

10. Expansion of an arbitrary function in Chebyshev polynomials. By taking into account the orthogonality property, it is an easy matter to show that if the series

$$2c_0 + \sum_{n=1}^{\infty} c_n T_n(x)$$

converges uniformly to the sum $f(x)$ in the interval $-1 \leqq x \leqq 1$, then

$$c_n = \frac{2}{\pi} \int_{-1}^{1} f(x) T_n(x) \frac{dx}{\sqrt{1 - x^2}}. \tag{30}$$

As an example of the use of a function expanded in Chebyshev polynomials, consider the curve of plate current versus grid voltage for a triode. (31) Linear theory of triode amplification is based on the assumption that the operating region is so small that the characteristic may be replaced by a straight line throughout the region with negligible error. This involves considering only the linear term in the power series expansion about the operating point. To calculate the harmonic distortion resulting from a finite grid swing, the higher derivative terms in the power series become important. From experimental measurement the coefficients of the Taylor series can only be derived from the limit towards which the measurements converge for an infinitely small grid swing.

Consider a sine wave applied to the grid, so that the grid voltage $e_g(t)$ is given by

$$e_g(t) = e_{g0} + a \cos \omega t. \tag{31}$$

The application of $e_g(t)$ causes a plate current to flow of the form

$$i_a(t) = I_0 + I_1 \cos \omega t + I_2 \cos 2\omega t + \cdots .$$

Now if we write

$$x = \cos \omega t \tag{32}$$

then

$$i_a(t) = I_0 + I_1 T_1(x) + I_2 T_2(x) + \cdots . \tag{33}$$

In other words, the amplitudes of the higher harmonics which are measured experimentally, are, for a given operating point and grid swing, the coefficients of the triode characteristic expanded in a series of Chebyshev polynomials. Thus the expression of the triode characteristic in this form is easy to obtain experimentally, and when once written, gives the information regarding the amount of harmonics under particular operating conditions.

Application of Chebyshev Polynomials to Antennas. It is frequently the problem in antenna design that the beam should be as narrow as possible, the power gain should be a maximum, and the sidelobes should be relatively small. These requirements are normally not easy or even possible to satisfy simultaneously, so the problem arises as to what to term the optimum field pattern. For this discussion we shall term that field pattern optimum which has a minimum beam width for a specified sidelobe level.

To determine a method for obtaining the optimum pattern, consider an array of n isotropic point sources of uniform spacing and all in the same phase, as shown in fig. 5. For even n, the array is given by 5a; for odd n, by 5b. The amplitudes of the individual sources are denoted by the A's as shown. Hence the total field for 5a in a direction θ is given, for large distances, by the expression (21)

$$E_{even} = 2A_1 \cos \phi/2 + 2A_2 \cos 3\phi/2 + \cdots + 2A_{n/2} \cos (n-1)\phi/2, \tag{34}$$

and for 5b,

$$E_{odd} = 2A_0 + 2A_1 \cos \phi + 2A_2 \cos 2\phi + \cdots + 2A_{(n-1)/2} \cos (n-1)\phi/2, \tag{35}$$

where $\phi = 2\pi d/\lambda \sin \theta$ and d is the distance between point sources. It is clear that (34)

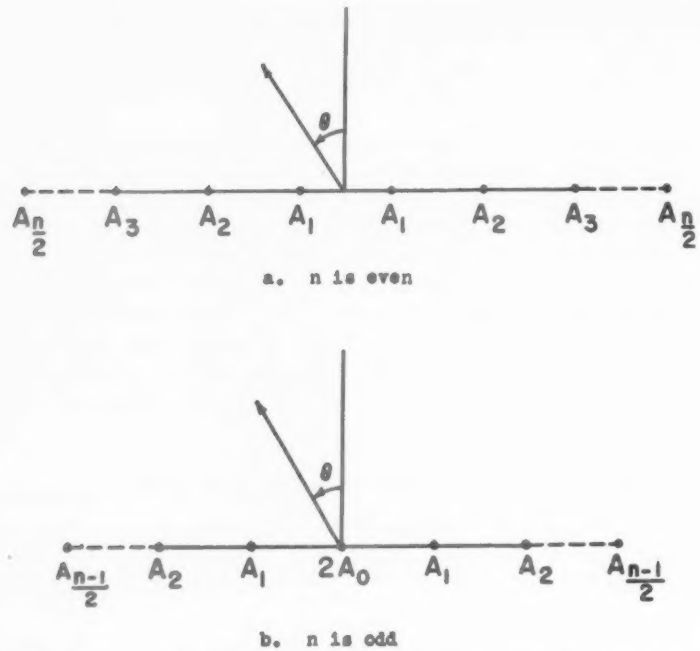

FIG. 5. Linear broadside array of n isotropic sources with uniform spacing. a. n is even. b. n is odd.

and (35) may be written in the form:

$$E_{even} = 2 \sum_{k=1}^{n/2} A_k \cos (2k - 1)\phi/2 = 2 \sum_{k=1}^{n/2} A_k T_{2k-1}(x), \tag{36}$$

$$E_{odd} = 2 \sum_{k=0}^{(n-1)/2} A_k \cos k\phi = 2 \sum_{k=0}^{(n-1)/2} A_k T_{2k}(x), \tag{37}$$

where $x = \cos \phi/2$.

It is now the problem to determine the value of the A's in (36) and (37) so that the optimum pattern is obtained. One way is to make all of the A's equal. This, indeed, gives a maximum gain and a narrow beam width, but the level of the side lobes is very high. Another way is to make the A's proportional to the binomial coefficients, in which case the side lobes disappear altogether if the spacing between elements is less than half a wave length, but unfortunately increased beam width and loss of gain result.

Now let us consider Dolph's method (14) of using the Chebyshev polynomials to obtain an optimum pattern. Suppose we have an array of n sources as in Fig. 5, where n may be odd or even. Let the specified ratio of the main lobe maximum to the minor lobe level be R. The problem is to find the relative source strengths for the pattern to be optimum in our prescribed sense.

The forms (36) and (37) show that the field due to the n given sources is described by a polynomial of $(n - 1)$th degree, which may be denoted by $P_{n-1}(x)$. The solution of the equation $P_{n-1}(x) = R$ will give x_R, the point at which the pattern assumes its

maximum value. Letting $y = x/x_R$, then $(1/R)\ P_{n-1}(y)$ will have a value of unity at $y = 1$, and we have that y (rather than x) is equal to $\cos \phi/2$. It is necessary that $P_{n-1}(y)$ have all its roots in the interval $(-1, 1)$, and furthermore be an even function about the origin. Thus $P_{n-1}(y)$ satisfies the requirements of Theorem 3. Hence it follows that if the distance from the greatest zero to the point $y = 1$ is to be a minimum (which must be the case for the beam width to be a minimum), then $P_{n-1}(y)$ must be proportional to the Chebyshev polynomial $T_{n-1}(x)$. Examples appear in Dolph's paper.

Relation of the Chebyshev polynomials to filters. Consider the network shown in fig. 6,

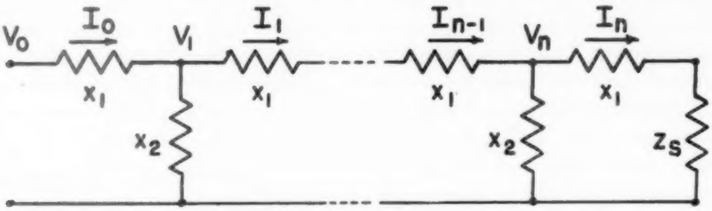

FIG. 6. Finite chain of reactive elements connected in ladder form.

which is composed of n identical circuits of isolated lossless elements. Let the driving emf at the input terminals be a sinusoidal source of amplitude V_0 and of angular frequency ω. Let x_1 and x_2 represent the reactances of the circuit elements at this frequency. Then

$$x_1 I_k = x_2(I_{k-1} + I_{k+1} - 2I_k)$$

or, by placing $x = x_1/x_2 + 2$,

$$I_{k+1} - xI_k + I_{k-1} = 0. \tag{38}$$

From equation (38) it can easily be shown that if

$$y_n(x) = x^n - \binom{n-1}{1}x^{n-2} + \binom{n-2}{2}x^{n-4} - \cdots$$

then in terms of I_0 an I_1, I_{k+1} may be expressed as

$$I_{k+1} = y_k I_1 - y_{k-1} I_0 . \tag{39}$$

In terms of I_0 and V_0:

$$V_k = -y_{k-1}x_1 I_0 + (y_{k-1} - y_{k-2})V_0 ,$$

$$I_k = (y_k - y_{k-1})I_0 - y_{k-1}V_0/x_2 .$$

The y's are polynomials satisfying the same recurrence relation as (38). By comparing the expansion of $y_n(x)$ with the expansion (22), the following relation between the y's and the Chebyshev polynomials can be derived:

$$y_n(x) = 2\left[T_n(x/2) + T_{n-2}(x/2) + \cdots \begin{array}{l} + T_1(x/2) \text{ if } n \text{ is odd.]} \\ + T_0(x/2) \text{ if } n \text{ is even.]} \end{array}\right. \tag{40}$$

Through the use of equation (40) not only can the successive currents and voltages in the network shown in fig. 6 be expressed in terms of Chebyshev polynomials, but also the input impedance and transfer impedance for any load z_s.

Application to directional couplers. It has been pointed out by F. Bolinder (2) that if one wishes to maximize the ratio of the width of the pass band to the level of the reverse wave in the pass band in a directional coupler, then for a given number of elements one will choose the coupling factors in such a way that the variation of the reversed current level with distance will be a Chebyshev polynomial.

Conclusion. This paper has presented the properties of the Chebyshev approximation method and the Chebyshev polynomials which may be of interest and use to engneers and physicists. The application of the method to filter design has not been covered here because of the length of the subject and the detail required. Glowatzki (16) gives tables for determining the number and sizes of elements needed in a filter which meets certain specified requirements. A number of other references on this subject are given in the bibliography, where also may be found references to articles of a detailed mathematical nature for those interested in this aspect of the problem:

Bibliography

[1] E. Borel, *Lecons sur les fonctions de variables, reeles et les developpements en series de polynomes*, Gauthier-Villars, Paris, 1905.

[2] F. Bolinder, *Approximate theory of the directional coupler*, Proc. IRE, **39**, 1951, 291.

[3] W. Cauer, *Ueber die Variabeln eines passiven Vierpols*, Sitzungsberichte der Preuss. Akad. der Wiss., 1927.

[4] W. Cauer, *Vierpole*, E.N.T., 1929.

[5] W. Cauer, *Siebschaltungen*, V.D.I., Verlag G.M.B.H. Berlin, 1931. (Condensed version in Physics, **2**, No. 4, 1932, 242-268.)

[6] W. Cauer, *Frequenzweichen konstanten Betriebswiderstandes*, E.N.T., Bd. 16, 1939, 96-120.

[7] J. Chokhatte, *Sur quelques proprietes des polynomes de Tchebycheff*, Comptes Rendus, **166**, 1918, 28-31.

[8] J. Chokhatte, *On a general formula in the theory of Tchebycheff polynomials and its applications*, Trans. Amer. Math. Soc., **29**, 1927, 569-583.

[9] J. Chokhatte, *Sur une formule general dans la theorie des polynomes de Tchebycheff et ses applications*, Comptes Rendus, **181**, 1925, 329-331.

[10] J. Chokhatte, *Sur quelques applications des polynomes de Tchebycheff a plusieurs variables*, Comptes Rendus, **183**, 1926, 442-444.

[11] J. Chokhatte, *Sur le developpement de l'integrale* $\int_a^b \frac{p(y)}{x-y}\,dy$ *en fraction continue et sur les polynomes de Tchebycheff*, Rendiconti del Circolo Matematico di Palermo, **47**, 1923, 25-46.

[12] A. Colombani, *La theorie des filtres electriques et les polynomes de Tchebichef*, J. de. Physique et le Radium, **7**, No. 8, 1946, 231-243.

[13] P. Coulombe, *Calcul rationnel des filtres en echelle*, Bull. Soc. Fran. des Elec., **6**, 1946, 103-110.

[14] C. L. Dolph, *A current distribution for broadside arrays which optimizes the relationship between beam width and side-lobe level*, Proc. IRE, **34**, 1946, 335-348.

[15] R. Feldtkeller, *Einfuhrung in die Theorie der Rundfunksiebschaltungen*, **3**, Auf., S. Hirzel, Leipzig, 1945.

[16] E. Glowatzki, *Entwurf und Beispiele symmetrischer Siebschaltungen nach der Methode von W. Cauer*, E.N.T., Teil I, Heft 9; Teil II, Heft 10; 1933.

[17] E. A. Guillemin, *A recent contribution to the design of electric filter networks*, Jnl. Math. and Phys., **11**, 1932, 150-211.

[18] E. S. Guillemin, *Communication networks*, II, John Wiley and Sons, New York, 1935.

[19] R. Julia, *Sur la theorie des filtres de W. Cauer*, Bull. Soc. Fran. des Elec., **5**, 1935, 983-1066.

[20] P. Kircherberger, *Uber Tchebychefsche Annaherungsmethoden*, Inaugural dissertation, Gottingen, 1932.

[21] J. D. Kraus, *Antennas*, McGraw-Hill, New York, 1950, 97-110.

[22] P. Le Corbeiller, *Methode d'approximation de Tchebychef et application aux filtres de frequences*, RGE, 40, 1936, 651-657.

[23] F. Leja, *Sur les polynomes de Tchebycheff et la fonction de Green*, Ann. Soc. Polon, Math., 19, 1946, 1-6.

[24] Stephan Lipka, *Uber die Anzahl der Nullstellen von Tschebyscheff Polynomen*, Monatsh. Math. Phys., 51, 1944, 173-178.

[25] R. L. Pritchard, *Optimum directivity patterns for linear arrays*, Tech. Memorandum #7, Acoustics Research Labs., Harvard, 1950, 3-5. Also, *Directivity of linear point arrays*, Ph.D. thesis, Harvard University.

[26] P. L. Tchebycheff, *Theorie des mecanismes connus sous le nom de parallelogrammes*, Oeuvres, 111-143.

[27] P. L. Tchebycheff, *Sur les questions de minima qui se rattachent a la representation approximatives des fonctions*, Mem. Acad. Imperiale des Sci. de St. Petersbourg, t. VII, 1859, 199-291. Also, Oeuvres, t. I, 273-378.

[28] P. L. Tchebycheff, *Sur les fonctions qui different le moins possible de zero*, Jnl. des Mathematiques pures et appliquees, t. XIX, 1874, 319-346. Oeuvres, t. II, 189-215.

[29] P. L. Tchebycheff, *Sur les fonctions qui s'ecartent peu de zero pour certaines valeurs de la variable*, Oeuvres, t. II, 335-356.

[30] B. van der Pol and Th. J. Weijers, *Tchebycheff polynomials and their relation to circular functions, Bessel functions, and Lissajous figures*, Physica, 1933, 1, 78-96.

[31] B. van der Pol and Th. J. Weijers, *Fine structure of triode characteristics*, Physica, 1, 1934, 481-496.

THE PLASTIC INSTABILITY OF PLATES*

BY

H. G. HOPKINS**

Manchester University

1. Introduction. A flow theory of the plastic instability of plates under edge stresses which can be used to provide information for structural design is not yet available. In this paper, an analysis of this problem, for rectangular plastes under uniform edge direct stresses, is given that provides information of a general nature only. In comparison with previous work by Handelman and Prager (1), Hopkins (2) and Pearson (3), the present paper exhibits both similarities and differences. In the first place, Refs. 1-3 and the present paper develop analyses within the framework of the elementary bending theory of plates and based upon a plastic flow theory relating to an idealized material whose mechanical behaviour is that of a work-hardening material obeying the Mises plasticity condition; further, in all these papers, the applied edge direct stress distributions are similar, always being uniform along an edge and taken to vanish for one pair of opposite edges in Refs. 1 and 3. In the second place, there is an implicit neglect of strain compatibility in the plate middle surface in Refs. 1-3. In brief, in comparison with the analyses of Refs. 1-3, the important features of the present analysis are that strain compatibility of the plate middle surface is included and that the development of the stress-rate/strain-rate relationships is more incisive. The present paper is based upon Ref. 4, and certain parts of the analysis developed there are here either curtailed or omitted for conciseness.

2. Notation. The length, breadth and thickness of the plate are respectively a, b and $2h$. The tensor notation employed in Section 3 is described there. In Section 4, the origin O is a point common to an edge of length b and the middle surface; the x- and y-axes are in the middle surface, and parallel to the plate edges; and the z-axis is such that the frame of reference $O(x, y, z)$ is right-handed. The displacement of a point during instability has the components hu, hv and hw respectively parallel to the x-, y-and z-axes. The notation for strains, stresses and plate resultants is standard except that reduced stresses, i.e. stresses divided by Young's modulus E_0, are used. The tangent modulus of elasticity is E, and Poisson's ratio (in the elastic range) is $\nu(0 \leq \nu \leq \frac{1}{2})$. The analysis is simplified with the use of the non-dimensional coordinates ξ, η and ζ defined by

$$\xi = x/a, \qquad \eta = y/a, \qquad \zeta = z/h, \tag{1}$$

and then, with an appropriate choice of the co-ordinate system, the origin now being taken at a corner of the plate, the unstrained plate occupies the region defined by

$$0 \leq \xi \leq 1, \qquad 0 \leq \eta \leq b/a, \qquad -1 \leq \zeta \leq 1. \tag{2}$$

Immediately before normal displacements occur, the plate is subject to the state of uniform normal edge stress defined by

$$\left.\begin{aligned} \sigma_x &= -\sigma_1(\xi = 0, 1; \quad 0 \leq \eta \leq b/a; \quad -1 \leq \zeta \leq 1), \\ \sigma_y &= -\sigma_2(\eta = 0, b/a; \quad 0 \leq \xi \leq 1; \quad -1 \leq \zeta \leq 1), \end{aligned}\right\} \tag{3}$$

*Received July 24, 1952.

**Now Visiting Professor of Applied Mathematics at Brown University.

where at least one of σ_1 and σ_2 is necessarily positive if normal displacement of the middle surface is to be possible. The following abbreviations are used:

$$
\left.
\begin{aligned}
\alpha &= a^2/b^2, \qquad \beta = h^2/a^2, \\
\sigma_1' &= 3(1 - \nu^2)\sigma_1/\beta, \qquad \sigma_2' = 3(1 - \nu^2)\sigma_2/\beta, \\
r_1 &= (2\sigma_1 - \sigma_2)/(\sigma_1 - 2\sigma_2), \qquad r_2 = (2\sigma_2 - \sigma_1)/(\sigma_2 - 2\sigma_1), \\
L &= -\left(1 - \frac{\nu}{r_1}\right)\left(1 - \frac{\nu}{r_2}\right) + \lambda_1\left(1 - \frac{\nu}{r_1}\right) + \lambda_2\left(1 - \frac{\nu}{r_2}\right) \\
&= (1 - \nu^2) + (\lambda_1 - 1)\{1 + 1/r_1^2 - 2\nu/r_1\} > 0, \\
c_1 &= (\lambda_1 - 1)/L(2\sigma_1 - \sigma_2), \qquad c_2 = (\lambda_2 - 1)/L(2\sigma_2 - \sigma_1), \\
M &= 1 - 2L \bigg/ \left\{\left(1 - \frac{\nu}{r_1}\right)(\lambda_1 - 1) + \left(1 - \frac{\nu}{r_2}\right)(\lambda_2 - 1)\right\} \\
&= -1 - 2(1 - \nu^2)/(\lambda_1 - 1)(1 + 1/r_1^2 - 2\nu/r_1) \leq -1, \\
N &= \{(1 - \nu/r_1)(\lambda_1 - 1) + (1 - \nu/r_2)(\lambda_2 - 1)\}/4(1 - \nu^2) \\
&= (\lambda_1 - 1)(1 + 1/r_1^2 - 2\nu/r_1)/4(1 - \nu^2) \geq 0, \\
d^* &= \tfrac{1}{4}(2 + \zeta_0)(1 - \zeta_0)^2, \\
\delta_1 &= -d^*(\lambda_1 - 1)(1 - \nu/r_1)/L, \qquad \delta_2 = -d^*(\lambda_2 - 1)(1 - \nu/r_2)/L, \\
D_{11} &= 1 + (1 - \nu/r_1)\delta_1\,, \\
D_{22} &= 1 + (1 - \nu/r_2)\delta_2\,, \\
2D_{12} &= 2 + (\nu - 1/r_1)\delta_1 + (\nu - 1/r_2)\delta_2\,.
\end{aligned}
\right\} \tag{4}
$$

In the analysis it is convenient to treat with rates-of-change of certain mathematical quantities rather than with their small changes, and primes generally denote time differentation.

3. Relationships Between the Stress- and Strain-Rates. In this Section, following Refs. (5) and (6), relationships are determined between stress- and strain-rates for a special type of work-hardening material; these relationships are developed within the framework of the elementary bending theory of thin plates and are appropriate to a rectangular plate, loaded initially only by uniform direct stresses along its edges, and in a configuration close to its initially flat configuration. Tensor notation is used in this Section; thus, if Ox_i denotes a right-handed rectangular cartesian frame of reference, then σ_{ij} denotes the stress tensor and ϵ_{ij}' denotes the strain-rate tensor corresponding to the stress-rate tensor σ_{ij}', where Latin suffixes have the values 1, 2 and 3, and the repeated suffix convention for summation is employed. For simplicity, reduced stresses and stress-rates, i.e. stresses and stress-rates divided by Young's modulus E_0, are employed.

The plastic strain-rate $\epsilon_{ij}^{\prime p}$ is defined by the equation

$$\epsilon_{ij}^{\prime p} = \epsilon_{ij}' - \epsilon_{ij}^{\prime e} , \tag{5}$$

where the elastic strain-rate $\epsilon_{ij}^{\prime e}$ is related to the stress-rate σ_{ij}' through Hooke's law, viz.

$$\epsilon_{ij}^{\prime e} = (1 + \nu)\sigma_{ij}' - \nu\sigma_{kk}'\delta_{ij} . \tag{6}$$

Then it is proved in Ref. 6 that the plastic strain-rate is given by the equation

$$\epsilon_{ij}^{\prime p} = G \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{kl}} \sigma_{kl}' , \tag{7}$$

where G is a non-zero scalar that may depend upon the current stress- and strain-states, together with their histories, but does not involve the current stress-rate, and where f is the loading function. Thus, from Eqs. (5)-(7), the stress-rate/strain-rate relationships are given by the equations,

$$\left.\begin{aligned} \epsilon_{ij}' &= (1 + \nu)\sigma_{ij}' - \nu\sigma_{kk}'\delta_{ij} + G \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{kl}} \sigma_{kl}' , & \frac{\partial f}{\partial \sigma_{kl}} \sigma_{kl}' > 0, \\ \epsilon_{ij}' &= (1 + \nu)\sigma_{ij}' - \nu\sigma_{kk}'\delta_{ij} , & \frac{\partial f}{\partial \sigma_{kl}} \sigma_{kl}' < 0. \end{aligned}\right\} \tag{8}$$

It is assumed now that the plastic strain-rates are incompressible so that

$$\epsilon_{kk}^{\prime p} = 0, \tag{9}$$

and from Eq. (7) it follows that

$$\frac{\partial f}{\partial \sigma_{kk}} \equiv \frac{\partial f}{\partial \sigma_{11}} + \frac{\partial f}{\partial \sigma_{22}} + \frac{\partial f}{\partial \sigma_{33}} = 0, \tag{10}$$

or, equivalently,

$$\frac{\partial f}{\partial \sigma} = 0 \qquad \text{where} \qquad \sigma = \frac{1}{3} \sigma_{kk} , \tag{11}$$

and hence the loading function is independent of σ and therefore depends only upon σ_{ij}^* where

$$\sigma_{ij}^* = \sigma_{ij} - \frac{1}{3} \sigma_{kk}\delta_{ij} . \tag{12}$$

If the co-ordinate axes $0x_i$ are suitably oriented then the initial state of stress in the plate, i.e. just before normal displacements occur, is represented by the tensor

$$\sigma_{ij} \equiv \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{13}$$

In the present connection, i.e. within the framework of the elementary bending theory of thin plates, the stress-rate tensor to which explicit attention is given when normal displacements first occur is represented by

$$\sigma_{ij}' \equiv \begin{bmatrix} \sigma_{11}' & \sigma_{12}' & 0 \\ \sigma_{21}' & \sigma_{22}' & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{14}$$

Therefore, from Eqs. (8) and (9), the stress-rate/strain-rate relationships, significant in the present connection, are given by the equations,

$$\left.\begin{array}{l}\left.\begin{array}{l}\epsilon'_{\alpha\beta} = (1+\nu)\sigma'_{\alpha\beta} - \nu\sigma'_{\gamma\gamma}\delta_{\alpha\beta} + G\dfrac{\partial f}{\partial\sigma_{\alpha\beta}}\dfrac{\partial f}{\partial\sigma_{\gamma\delta}}\sigma'_{\gamma\delta}\,, \\ \epsilon'_{33} = -\nu\sigma'_{\gamma\gamma} - G\left(\dfrac{\partial f}{\partial\sigma_{11}} + \dfrac{\partial f}{\partial\sigma_{22}}\right)\dfrac{\partial f}{\partial\sigma_{\gamma\delta}}\sigma'_{\gamma\delta}\,,\end{array}\right\}\dfrac{\partial f}{\partial\sigma_{\gamma\delta}}\sigma'_{\gamma\delta} > 0, \\ \left.\begin{array}{l}\epsilon'_{\alpha\beta} = (1+\nu)\sigma'_{\alpha\beta} - \nu\sigma'_{\gamma\gamma}\delta_{\alpha\beta}\,, \\ \epsilon'_{33} = -\nu\sigma'_{\gamma\gamma}\,,\end{array}\right\}\dfrac{\partial f}{\partial\sigma_{\gamma\delta}}\sigma'_{\gamma\delta} < 0,\end{array}\right\} \tag{15}$$

where Greek suffixes have the values 1 and 2, and the repeated suffix convention for summation is employed. It is reasonable to assume that the direct stress-rates produce zero or negligible shear strain-rates, and then, from Eqs. (15),

$$\frac{\partial f}{\partial\sigma_{12}} = 0 \qquad \text{for} \qquad \sigma_{12} = \sigma_{33} = \sigma_{13} = 0. \tag{16}$$

Eqs. (15) now simplify to the equations,

$$\left.\begin{array}{l}\left.\begin{array}{l}\epsilon'_{11} = \sigma'_{11} - \nu\sigma'_{22} \\ \qquad + G\dfrac{\partial f}{\partial\sigma_{11}}\left(\dfrac{\partial f}{\partial\sigma_{11}}\sigma'_{11} + \dfrac{\partial f}{\partial\sigma_{22}}\sigma'_{22}\right), \\ \epsilon'_{12} = (1+\nu)\sigma'_{12}\,, \\ \epsilon'_{22} = -\nu\sigma'_{11} + \sigma'_{22} \\ \qquad + G\dfrac{\partial f}{\partial\sigma_{22}}\left(\dfrac{\partial f}{\partial\sigma_{11}}\sigma'_{11} + \dfrac{\partial f}{\partial\sigma_{22}}\sigma'_{22}\right), \\ \epsilon'_{33} = -\nu(\sigma'_{11} + \sigma'_{22}) \\ \qquad - G\left(\dfrac{\partial f}{\partial\sigma_{11}} + \dfrac{\partial f}{\partial\sigma_{22}}\right)\left(\dfrac{\partial f}{\partial\sigma_{11}}\sigma'_{11} + \dfrac{\partial f}{\partial\sigma_{22}}\sigma'_{22}\right),\end{array}\right\}\dfrac{\partial f}{\partial\sigma_{11}}\sigma'_{11} + \dfrac{\partial f}{\partial\sigma_{22}}\sigma'_{22} > 0, \\ \left.\begin{array}{l}\epsilon'_{11} = \sigma'_{11} - \nu\sigma'_{22}\,, \\ \epsilon'_{12} = (1+\nu)\sigma'_{12}\,, \\ \epsilon'_{22} = -\nu\sigma'_{11} + \sigma'_{22}\,, \\ \epsilon'_{33} = -\nu(\sigma'_{11} + \sigma'_{22}),\end{array}\right\}\dfrac{\partial f}{\partial\sigma_{11}}\sigma'_{11} + \dfrac{\partial f}{\partial\sigma_{22}}\sigma'_{22} < 0.\end{array}\right\} \tag{17}$$

Now define two scalars λ_1 and λ_2 through the equations

$$\left.\begin{aligned} \lambda_1 &= \epsilon'_{11}/\sigma'_{11}, \qquad \sigma'_{22} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial \sigma_{11}}\,\sigma'_{11} > 0,\\ \lambda_2 &= \epsilon'_{22}/\sigma'_{22}, \qquad \sigma'_{11} = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial \sigma_{22}}\,\sigma'_{22} > 0,\end{aligned}\right\} \tag{18}$$

and then, from Eqs. (17),

$$\lambda_1 = 1 + G\left(\frac{\partial f}{\partial \sigma_{11}}\right)^2, \qquad \lambda_2 = 1 + G\left(\frac{\partial f}{\partial \sigma_{22}}\right)^2, \tag{19}$$

and hence

$$G^{1/2}\,\frac{\partial f}{\partial \sigma_{11}} = \pm(\lambda_1 - 1)^{1/2}, \qquad G^{1/2}\,\frac{\partial f}{\partial \sigma_{22}} = \pm(\lambda_2 - 1)^{1/2}, \tag{20}$$

where, in the first equation, the positive and negative signs correspond respectively to loading and unloading in the special case $\sigma'_{11} > 0$ with $\sigma'_{22} = 0$, and similarly for the second of these equations. Therefore, from Eqs. (17) and (20), it is possible to formulate stress-rate/strain-rate relationships that involve the two scalars λ_1 and λ_2 whose numerical values could be determined from experimental data. Note that

$$(\lambda_1 - 1)/(\lambda_2 - 1) = \left(\frac{\partial f}{\partial \sigma_{11}}\bigg/\frac{\partial f}{\partial \sigma_{22}}\right)^2. \tag{21}$$

The simplest possible assumption for f is that the Mises plasticity condition applies, viz.

$$f = \frac{1}{2}\,\sigma^*_{ij}\sigma^*_{ij} = k^2, \tag{22}$$

where the strain-history enters only through the single parameter k. Loading occurs only if

$$f' = \sigma^*_{ij}\sigma'^*_{ij} > 0, \tag{23}$$

and in the present connection the criterion is therefore that

$$\begin{aligned}&\left\{\sigma_{11} - \frac{1}{3}(\sigma_{11} + \sigma_{22})\right\}\left\{\sigma'_{11} - \frac{1}{3}(\sigma'_{11} + \sigma'_{22})\right\}\\ &\quad + \left\{\sigma_{22} - \frac{1}{3}(\sigma_{11} + \sigma_{22})\right\}\left\{\sigma'_{22} - \frac{1}{3}(\sigma'_{11} + \sigma'_{22})\right\} + \frac{1}{9}(\sigma_{11} + \sigma_{22})(\sigma'_{11} + \sigma'_{22}) > 0,\end{aligned} \tag{24}$$

i.e.,

$$f' = \frac{1}{3}\,\{(2\sigma_{11} - \sigma_{22})\sigma'_{11} + (2\sigma_{22} - \sigma_{11})\sigma'_{22}\} > 0. \tag{25}$$

Thus

$$\frac{\partial f}{\partial \sigma_{11}} = \frac{1}{3}(2\sigma_{11} - \sigma_{22}), \qquad \frac{\partial f}{\partial \sigma_{22}} = \frac{1}{3}(2\sigma_{22} - \sigma_{11}), \tag{26}$$

and accordingly, from Eq. (21),

$$(\lambda_1 - 1)/(\lambda_2 - 1) = \{(2\sigma_{11} - \sigma_{22})/(2\sigma_{22} - \sigma_{11})\}^2. \tag{27}$$

From Eqs. (19) and (26),

$$G = 9(\lambda_1 - 1)/(2\sigma_{11} - \sigma_{22})^2 = 9(\lambda_2 - 1)/(2\sigma_{22} - \sigma_{11})^2, \tag{28}$$

and G is calculable if either λ_1 or λ_2 is known from experimental data. It is possible therefore to formulate the stress-rate/strain-rate relationships in terms of a single scalar to be determined necessarily from experimental data; these relationships are

$$\left.\begin{array}{l}
\left.\begin{array}{l}
\epsilon'_{11} = \lambda_1\sigma'_{11} - \left\{\nu + \dfrac{1}{R_1}(\lambda_1 - 1)\right\}\sigma'_{22}\,, \\[2ex]
\epsilon'_{12} = (1 + \nu)\sigma'_{12}\,, \\[2ex]
\epsilon'_{22} = -\left\{\nu + \dfrac{1}{R_2}(\lambda_2 - 1)\right\}\sigma'_{11} + \lambda_2\sigma'_{22}\,, \\[2ex]
\epsilon'_{33} = -\left\{\nu + (\lambda_1 - 1) - \dfrac{1}{R_2}(\lambda_2 - 1)\right\}\sigma'_{11} \\[2ex]
\qquad\quad -\left\{\nu + (\lambda_2 - 1) - \dfrac{1}{R_1}(\lambda_1 - 1)\right\}\sigma'_{22}\,,
\end{array}\right\} (2\sigma_{11} - \sigma_{22})\sigma'_{11} + (2\sigma_{22} - \sigma_{11})\sigma'_{22} > 0, \\[6ex]
\left.\begin{array}{l}
\epsilon'_{11} = \sigma'_{11} - \nu\sigma'_{22}\,, \\[2ex]
\epsilon'_{12} = (1 + \nu)\sigma'_{12}\,, \\[2ex]
\epsilon'_{22} = -\nu\sigma'_{11} + \sigma'_{22}\,, \\[2ex]
\epsilon'_{33} = -\nu(\sigma'_{11} + \sigma'_{22}),
\end{array}\right\} (2\sigma_{11} - \sigma_{22})\sigma'_{11} + (2\sigma_{22} - \sigma_{11})\sigma'_{22} < 0,
\end{array}\right\} \tag{29}$$

where

$$R_1 = (2\sigma_{11} - \sigma_{22})/(\sigma_{11} - 2\sigma_{22}), \qquad R_2 = (2\sigma_{22} - \sigma_{11})/(\sigma_{22} - 2\sigma_{11}), \tag{30}$$

and λ_1 and λ_2 are related through Eq. (27). These stress-rate/strain-rate relationships agree with those obtained in Refs. 1 and 2. However, in Ref. 2, the result Eq. (27) could be deduced only for the particular case discussed in Ref. 1, viz. σ_{22} equal to zero, and the result is then

$$\lambda_2 = \frac{1}{4}(\lambda_1 + 3), \tag{31}$$

where λ_1 is interpreted simply as E_0/E.

In the next Section, it is more convenient to use an engineering notation for stresses

and strains, and accordingly Eqs. (29) are rewritten now in the form,

$$\left.\begin{aligned}
&\left.\begin{aligned}
\epsilon_x' &= \lambda_1\sigma_x' - \left\{\nu + \frac{1}{r_1}(\lambda_1 - 1)\right\}\sigma_y',\\
\gamma_{xy}' &= 2(1+\nu)\tau_{xy}',\\
\epsilon_y' &= -\left\{\nu + \frac{1}{r_2}(\lambda_2 - 1)\right\}\sigma_x' + \lambda_2\sigma_y',\\
\epsilon_z' &= -\left\{\nu + (\lambda_1 - 1) - \frac{1}{r_2}(\lambda_2 - 1)\right\}\sigma_x'\\
&\quad -\left\{\nu + (\lambda_2 - 1) - \frac{1}{r_1}(\lambda_1 - 1)\right\}\sigma_y',
\end{aligned}\right\}(2\sigma_1 - \sigma_2)\sigma_x' + (2\sigma_2 - \sigma_1)\sigma_y' < 0,\\
&\left.\begin{aligned}
\epsilon_x' &= \sigma_x' - \nu\sigma_y',\\
\gamma_{xy}' &= 2(1+\nu)\tau_{xy}',\\
\epsilon_y' &= -\nu\sigma_x' + \sigma_y',\\
\epsilon_z' &= -\nu(\sigma_x' + \sigma_y'),
\end{aligned}\right\}(2\sigma_1 - \sigma_2)\sigma_x' + (2\sigma_2 - \sigma_1)\sigma_y' > 0.
\end{aligned}\right\} \tag{32}$$

Let the rate at which work is done by existing stresses on the change of shape of an element be W' per unit volume so that

$$W' = E_0\sigma_{ij}\epsilon_{ij}'^{*}\,. \tag{33}$$

It may be proved from Eqs. (8) and (22) that

$$W' = \begin{cases} [(1+\nu) + 2Gk^2]E_0f', & f' > 0,\\ (1+\nu)E_0f', & f' < 0. \end{cases} \tag{34}$$

Thus the loading condition that f' is positive is interpreted that W' is positive; the neutral condition that f' is zero is interpreted that W' is zero; and the unloading condition that f' is negative is interpreted that W' is negative. From Eqs. (13) and (32) the loading and unloading criteria are now written in the form

$$\left.\begin{gathered}
\{(2\sigma_2 - \sigma_1)\nu + (2\sigma_1 - \sigma_2)\}\epsilon_x' + \{(2\sigma_1 - \sigma_2)\nu + (2\sigma_2 - \sigma_1)\}\epsilon_y' \lesseqgtr 0,\\
W' \gtrless 0,
\end{gathered}\right\} \tag{35}$$

respectively.

4. The Fundamental Equations. In this Section, the fundamental equations are developed on the basis of the stress-rate/strain-rate relationships in Eqs. (32). It is assumed, exactly as in elastic thin plate theory, that plate elements originally normal to the middle

surface remain normal to the displaced middle surface. Then

$$\left.\begin{aligned} \epsilon_x' &= \epsilon_1' - \beta\zeta w'_{,\xi\xi} , \\ \epsilon_y' &= \epsilon_2' - \beta\zeta w'_{,\eta\eta} , \\ \gamma_{xy}' &= \gamma' - 2\beta\zeta w'_{,\xi\eta} , \end{aligned}\right\} \tag{36}$$

where ϵ_1' , ϵ_2' and γ' are strain-rates in the middle surface. Here, and also elsewhere in this paper, suffixes ξ and η (and also n and s) preceded by a comma denote partial differential coefficients. From condition (35) and Eqs. (36), the loading criterion is

$$\zeta \gtrless \zeta_0 \qquad \text{if} \qquad K' \gtrless 0 \tag{37}$$

respectively, where

$$\left.\begin{aligned} \beta\zeta_0 K' &= (2\sigma_1 - \sigma_2)\{(1 - \nu/r_1)\epsilon_1' + (\nu - 1/r_1)\epsilon_2'\}, \\ K' &= (2\sigma_1 - \sigma_2)\{(1 - \nu/r_1)w'_{,\xi\xi} + (\nu - 1/r_1)w'_{,\eta\eta}\}. \end{aligned}\right\} \tag{38}$$

It is necessary always that $-1 \leq \zeta_0 \leq 1$; if $-1 < \zeta_0 < 1$, then the surface with equation $\zeta = \zeta_0$ separates loading and unloading regions, and on this surface $W' = 0$; $\zeta_0 = \pm 1$ corresponds to either loading or unloading over the entire plate thickness, and on the (plate) surface $\zeta = \pm 1$, in general $W' \neq 0$, and therefore this surface is not in this sense a boundary to a loading or an unloading region. It is seen by setting either $w'_{,\xi\xi}$ or $w'_{,\eta\eta}$ equal to zero that the sign of K' is an indication of the sense of bending.

The resultant force-rates N_x' , N_y' and N_{xy}' are defined by the equations

$$(N_x' , N_y' , N_{xy}')/h = \int_{-1}^{1} (\sigma_x' , \sigma_y' , \tau_{xy}') \, d\zeta, \tag{39}$$

and therefore

$$(N_x' - \nu N_y' , N_y' - \nu N_x')/h = \int_{-1}^{1} (\sigma_x' - \nu\sigma_y' , \sigma_y' - \nu\sigma_x') \, d\zeta. \tag{40}$$

From Eqs. (32) and (38) and conditions (37),

$$\left.\begin{aligned} \sigma_x' - \nu\sigma_y' &= \epsilon_x' + \beta c_1(\zeta - \zeta_0)K', \\ \sigma_y' - \nu\sigma_x' &= \epsilon_y' + \beta c_2(\zeta - \zeta_0)K', \\ \tau_{xy}' &= \gamma_{xy}'/2(1 + \nu), \end{aligned}\right\} \qquad (W' > 0), \tag{41}$$

and

$$\left.\begin{aligned} \sigma_x' - \nu\sigma_y' &= \epsilon_x' , \\ \sigma_y' - \nu\sigma_x' &= \epsilon_y' , \\ \tau_{xy}' &= \gamma_{xy}'/2(1 + \nu), \end{aligned}\right\} \qquad (W' < 0). \tag{42}$$

Conditions (37) show that there are two cases to be considered and the analysis now proceeds differently according as $K' > 0$ (Case A) or $K' < 0$ (Case B).

Case A. Here $K' > 0$, and then $(1 \geq \zeta > \zeta_0)$ and $(\zeta_0 > \zeta \geq -1)$ are respectively the loading and unloading regions.

From Eqs. (36) and (39)-(42),

$$\left.\begin{aligned} (N'_x - \nu N'_y)/2h &= \epsilon'_1 + \tfrac{1}{4}\beta c_1(1 - \zeta_0)^2 K', \\ (N'_y - \nu N'_x)/2h &= \epsilon'_2 + \tfrac{1}{4}\beta c_2(1 - \zeta_0)^2 K', \\ N'_{xy}/2h &= \gamma'/2(1 + \nu), \end{aligned}\right\} \tag{43}$$

and therefore

$$-\tfrac{1}{4}\beta(1 - \zeta_0)^2 K' = \{\epsilon'_1 - (N'_x - \nu N'_y)/2h\}/c_1 = \{\epsilon'_2 - (N'_y - \nu N'_x)/2h\}/c_2 . \tag{44}$$

From previous remarks, and from Eqs. (38.1) and (44), $\zeta_0 = \zeta_1$ if $-1 \leq \zeta_1 \leq 1$ and $\zeta_0 = \pm 1$ otherwise where $\zeta_1 = \zeta^*(P)$ is defined by the equation

$$\zeta^{*2} - 2M\zeta^* + P = 0, \tag{45}$$

i.e.

$$\zeta^* = M \pm (M^2 - P)^{1/2}, \tag{46}$$

and

$$\left.\begin{aligned} P = 1 - 4L\{(2\sigma_1 - \sigma_2)(1 - \nu/r_1)p_1 + (2\sigma_2 - \sigma_1)(1 - \nu/r_2)p_2\} \\ \div \{(\lambda_1 - 1)(1 - \nu/r_1) + (\lambda_2 - 1)(1 - \nu/r_2)\}, \\ p_1 = (N'_x - \nu N'_y)/2h\beta K', \\ p_2 = (N'_y - \nu N'_x)/2h\beta K'. \end{aligned}\right\} \tag{47}$$

If $-1 \leq \zeta^*(P) \leq 1$, where $\zeta^*(P)$ is defined by Eq. (46), then $\zeta_0 = \zeta^*(P)$ and the surface with equation $\zeta = \zeta_0 = \zeta^*(P)$ separates loading and unloading regions of the plate. If such a surface of separation exists in the immediate neighbourhood of some particular point of the middle surface, then, at this point, it is necessary first that

$$M^2 \geq P \tag{48}$$

and second that it is then possible to choose the sign in Eq. (46) so that

$$-1 \leq \zeta^*(P) \leq 1; \tag{49}$$

if either of the conditions (48) and (49) is violated, then loading or unloading occurs over the whole of the plate thickness in the immediate neighbourhood of this point, and then $\zeta_0 = \pm 1$. It is therefore not impossible for loading (or unloading) to occur over the entire thickness of part of the plate and a varying degree of loading (or unloading) to occur over the remaining part. Note that both roots of Eq. (45) may satisfy $-1 \leq \zeta^*(P) \leq 1$ in which case there are two possible solutions.

The resultant bending and twisting moment-rates M'_x, M'_y and $M'_{xy}(= -M'_{yx})$ are defined by the equations

$$(M'_x , M'_y , M'_{xy})/h^2 = \int_{-1}^{1} (\sigma'_x , \sigma'_y , -\tau'_{xy})\zeta \, d\zeta, \tag{50}$$

and therefore

$$(M_x' - \nu M_y' , M_y' - \nu M_x')/h^2 = \int_{-1}^{1} (\sigma_x' - \nu\sigma_y' , \sigma_y' - \nu\sigma_x')\zeta \, d\zeta. \tag{51}$$

From Eqs. (36), (41), (42), (50) and (51),

$$\left.\begin{aligned} M_x' - \nu M_y' &= -\tfrac{2}{3}h^2\beta(w'_{,\xi\xi} - dc_1K'), \\ M_y' - \nu M_x' &= -\tfrac{2}{3}h^2\beta(w'_{,\eta\eta} - dc_2K'), \\ M_{xy}' = -M_{yx}' &= \{\tfrac{2}{3}h^2\beta/(1 - \nu^2)\}(1 - \nu)w'_{,\xi\eta} , \end{aligned}\right\} \tag{52}$$

where $d = d^*(\zeta_0)$, and therefore

$$\left.\begin{aligned} -M_x'/\{\tfrac{2}{3}h^2\beta/(1 - \nu^2)\} &= \{1 + \delta_1(1 - \nu/r_1)\}w'_{,\xi\xi} + \{\nu + \delta_2(\nu - 1/r_2)\}w'_{,\eta\eta} , \\ -M_y'/\{\tfrac{2}{3}h^2\beta/(1 - \nu^2)\} &= \{\nu + \delta_1(\nu - 1/r_1)\}w'_{,\xi\xi} + \{1 + \delta_2(1 - \nu/r_2)\}w'_{,\eta\eta} , \\ M_{xy}'/\{\tfrac{2}{3}h^2\beta/(1 - \nu^2)\} &= (1 - \nu)w'_{,\xi\eta} . \end{aligned}\right\} \tag{53}$$

The resultant total vertical shear force-rates V_x' and V_y' are defined by the equations

$$\left.\begin{aligned} aV_x' &= M_{x,\xi}' - 2M_{xy,\eta}' , \\ aV_y' &= M_{y,\eta}' - 2M_{xy,\xi}' , \end{aligned}\right\} \tag{54}$$

and therefore from Eqs. (53)

$$\left.\begin{aligned} -V_x'/\{\tfrac{2}{3}a\beta^2/(1 - \nu^2)\} &= \frac{\partial}{\partial\xi}[\{1 + \delta_1(1 - \nu/r_1)\}w'_{,\xi\xi} + \{(2 - \nu) + \delta_2(\nu - 1/r_2)\}w'_{,\eta\eta}], \\ -V_y'/\{\tfrac{2}{3}a\beta^2/(1 - \nu^2)\} &= \frac{\partial}{\partial\eta}[\{1 + \delta_2(1 - \nu/r_2)\}w'_{,\eta\eta} + \{(2 - \nu) + \delta_1(\nu - 1/r_1)\}w'_{,\xi\xi}]. \end{aligned}\right\} \tag{55}$$

Now consider the equilibrium of the plate when small normal displacements have occurred. The equations of equilibrium (7, § 57) of the plate, for small normal displacements of its middle surface, are

$$\left.\begin{aligned} M_{x,\xi\xi} - 2M_{xy,\xi\eta} + M_{y,\eta\eta} &= -h(N_xw_{,\xi\xi} + 2N_{xy}w_{,\xi\eta} + N_yw_{,\eta\eta}), \\ N_{x,\xi} + N_{xy,\eta} &= 0, \\ N_{xy,\xi} + N_{y,\eta} &= 0; \end{aligned}\right\} \tag{56}$$

partial differentiation of Eqs. (56) with respect to time shows that

$$\left.\begin{aligned} M_{x,\xi\xi}' - 2M_{xy,\xi\eta}' + M_{y,\eta\eta}' &= 2h^2(\sigma_1w'_{,\xi\xi} + \sigma_2w'_{,\eta\eta}), \\ N_{x,\xi}' + N_{xy,\eta}' &= 0, \\ N_{xy,\xi}' + N_{y,\eta}' &= 0, \end{aligned}\right\} \tag{57}$$

immediately following the incidence of normal displacements because immediately

before these occur w and N_{xy} are identically zero. From Eqs. (53) and (57.1),

$$\nabla_1^4 w' + (1 - \nu/r_1)\frac{\partial^2}{\partial\xi^2}(\delta_1 w'_{,\xi\xi}) + (\nu - 1/r_2)\frac{\partial^2}{\partial\xi^2}(\delta_2 w'_{,\eta\eta})$$

$$+ (\nu - 1/r_1)\frac{\partial^2}{\partial\eta^2}(\delta_1 w'_{,\xi\xi}) + (1 - \nu/r_2)\frac{\partial^2}{\partial\eta^2}(\delta_2 w'_{,\eta\eta}) = -(\sigma_1' w'_{,\xi\xi} + \sigma_2' w'_{,\eta\eta}), \tag{58}$$

where

$$\nabla_1^4 \equiv \left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)^2. \tag{59}$$

Equations (57.2) and (57.3) are satisfied identically through the introduction of a stress-rate function φ' that is related to the resultant force-rates by the equations,

$$N_x'/2h = \varphi'_{,\eta\eta}, \qquad N_y'/2h = \varphi'_{,\xi\xi}, \qquad N_{xy}'/2h = -\varphi'_{,\xi\eta}. \tag{60}$$

The strain-rates ϵ_x', ϵ_y' and γ_{xy}' are given by the equations

$$\epsilon_x' = \beta^{1/2}u'_{,\xi}, \qquad \epsilon_y' = \beta^{1/2}v'_{,\eta}, \qquad \gamma_{xy}' = \beta^{1/2}(u'_{,\eta} + v'_{,\xi}), \tag{61}$$

and are compatible only if

$$\epsilon'_{x,\eta\eta} + \epsilon'_{y,\xi\xi} - \gamma'_{xy,\xi\eta} = 0, \tag{62}$$

i.e. from Eqs. (36),

$$\epsilon'_{1,\eta\eta} + \epsilon'_{2,\xi\xi} - \gamma'_{,\xi\eta} = 0, \tag{63}$$

which is an equation that involves the middle surface strain-rates only. From Eqs. (44) and (60),

$$\left.\begin{aligned} \epsilon_1' &= \varphi'_{,\eta\eta} - \nu\varphi'_{,\xi\xi} - \tfrac{1}{4}\beta c_1(1 - \zeta_0)^2 K', \\ \epsilon_2' &= \varphi'_{,\xi\xi} - \nu\varphi'_{,\eta\eta} - \tfrac{1}{4}\beta c_2(1 - \zeta_0)^2 K', \\ \gamma' &= -2(1 + \nu)\varphi'_{,\xi\eta}, \end{aligned}\right\} \tag{64}$$

and therefore Eq. (63) is

$$\nabla_1^4\varphi' = \frac{1}{4}\beta\left(c_2\frac{\partial^2}{\partial\xi^2} + c_1\frac{\partial^2}{\partial\eta^2}\right)\{(1 - \zeta_0)^2 K'\}. \tag{65}$$

Case B. Here $K' < 0$, and then $(1 \geq \zeta > \zeta_0)$ and $(\zeta_0 > \zeta \geq -1)$ are respectively the loading and unloading regions. It is straightforward to show that the analysis of Case B is deducible from the analysis of Case A if all quantities are now referred to the coordinate system $O'(\xi', \eta', \zeta')$ where

$$\xi' = \text{const.} - \xi, \qquad \eta = \text{const.} - \eta, \qquad \zeta' = -\zeta, \tag{66}$$

the values of the constants being irrelevant. Full details are given in Ref. 4, and here only the results are stated. The equations that correspond to Eqs. (53), (55) and (58) are identical in form but now $d = d^*(-\zeta_0)$ and $\zeta_1 = -\zeta^*(-p_1, -p_2)$; the equations that correspond to Eqs. (60) are identical in form; and the equation that corresponds to Eq. (65) is

$$\nabla_1^4\varphi' = -\frac{1}{4}\beta\left(c_2\frac{\partial^2}{\partial\xi^2} + c_1\frac{\partial^2}{\partial\eta^2}\right)\{(1 + \zeta_0)^2 K'\} \tag{67}$$

where again $\zeta_1 = -\zeta^*(-p_1, -p_2)$. The convention is now adopted that the above changes are implied if $K' < 0$, and then Eqs. (53), (55,) (58), (60) and (65) apply unrestrictedly.

The fundamental mathematical problem is therefore to solve the non-linear, dependent, fourth order, homogeneous, partial differential equations (58) and (65) subject to boundary conditions on w' and φ'. These boundary conditions depend upon respectively the plate edge support and the applied resultant force-rates, and, in general, are not arbitrarily prescribed. If the middle surface is sub-divided into two or more regions by a curve c, say with normal direction n, on which $K' = 0$, then it is necessary on physical grounds that w' and φ', together with their first three partial differential coefficients with respect to n, are continuous on c; but, from Eqs. (58) and (60), $w'_{,nnnn}$ and $\varphi'_{,nnnn}$ are not, in general, continuous on c and therefore in general the analytical form of the solution is not the same on either side of c.

Now consider the application of the analysis to two particular classes of problem. These are often designated in the published literature as the von Kármán and Shanley problems; in this paper they are designated briefly as problems A and B, respectively.

Let n and s denote directions respectively normal to, and along, a plate edge.

Problem A. Here it is postulated that the changes of the applied edge stresses at instability are zero. Then, by definition of the problem,

$$\varphi'_{,ss} = \varphi'_{,ns} = 0 \tag{68}$$

along an edge, but note that, because Eq. (65) is not homogeneous in φ', Eqs. (68) do not imply that φ' can be taken as identically zero. The boundary conditions on w' depend upon the plate edge support; for example, if the edges are either clamped or simply-supported (which are important practical cases) then the boundary conditions on w' are respectively either

$$w' = w'_{,n} = 0 \tag{69}$$

or, from Eqs. (53),

$$w' = w'_{,nn} = 0$$

along an edge. The solution of Eqs. (58) and (65) subject to the boundary conditions (68) and, for example, (69) is clearly, in general, an extremely difficult problem. If the instability stresses do not greatly exceed the elastic limit, then it is not unreasonable to approximate Eq. (65) by the equation

$$\nabla_1^4\varphi' = 0. \tag{70}$$

This approximation is made implicitly in Refs. 1 and 2, and results in a considerable simplification of the problem. The appropriate solution of Eq. (70) subject to the conditions (68) is

$$\varphi' = 0. \tag{71}$$

Then, from Eqs. (46) and (47), remembering that M is not greater than -1,

$$\zeta^* = M + (M^2 - 1)^{1/2} \tag{72}$$

and this equation necessarily gives a value for ζ^* such that $-1 \leq \zeta^* \leq 1$ and therefore $\zeta_0 = \pm\zeta^*$ according as $K' \gtrless 0$; there is now a surface separating loading and unloading

regions and it has the equation $\zeta = \pm \zeta^*$ according as $K' \gtrless 0$, i.e. a surface composed of parts of the planes $\zeta = \pm \zeta^*$ with discontinuities at points such that $K' = 0$. Such discontinuities may be due to the approximate nature of the solution. Further $d = d^*(\pm\zeta_0)$ according as $K' \gtrless 0$ and therefore $d = d^*(\zeta^*)$ in both cases. Equation (58) is now

$$D_{11}w'_{.\xi\xi\xi\xi} + 2D_{12}w'_{.\xi\xi\eta\eta} + D_{22}w'_{.\eta\eta\eta\eta} = -(\sigma_1' w'_{.\xi\xi} + \sigma_2' w'_{.\eta\eta}) \tag{73}$$

where D_{11} , D_{22} and D_{12} (see Eqs. (4)) are constants dependent upon the stress-state. The problem therefore depends upon the solution of the linear, fourth order, homogeneous, partial differential equation (73) with constant coefficients dependent upon the stress-state, subject to boundary conditions on w', e.g. Eqs. (69); the characteristic-values and characteristic-functions of this equation correspond respectively to the instability stresses and normal displacement-rates. Note that, although Eq. (73) has a doubly-infinite set of characteristic-values (and corresponding characteristic-functions), only the first is of importance from a practical standpoint, and further only in this case may Eq. (70) be a useful approximation to Eq. (65). Equation (73) is given in Ref. 2 and also, for the particular case σ_2 equal to zero, in Ref. 2. As remarked in Ref. 2, there is no serious mathematical difficulty involved in the determination of the (approximate) instability stresses for the edge conditions that are of practical importance; indeed the only difficulties are in the computation, and it is useful to note that it is simplest to regard σ_1 , σ_2 and α (or β) as known and to solve for β (or α).

Problem B. Here it is postulated that the changes of the applied edge stresses at instability are such that plastic deformation only occurs.

It is evident on physical grounds that, if it is possible for loading, simulateously with bending, to occur over the whole of the plate, then the plate bending stiffness is least and therefore so also are the instability stresses. An analysis based on this assumption, in any case, must lead necessarily to a lower bound for instability stresses; the assumption is discussed later.

Then, by definition of the problem, $\zeta_0 = \mp 1$ according as $K' \gtrless 0$; notice, however, that the (plate) surface $\zeta = \pm 1$, in general, is not a boundary to a loading or an unloading region. Further $d = d^*(-1) = 1$ in both cases. Equation (58) is now

$$D_{11}w'_{.\xi\xi\xi\xi} + 2D_{12}w'_{.\xi\xi\eta\eta} + D_{22}w'_{.\eta\eta\eta\eta} = -(\sigma_1' w'_{.\xi\xi} + \sigma_2' w'_{.\eta\eta}) \tag{74}$$

where D_{11} , D_{22} and D_{12} (given from Eqs. (4) after setting $d = 1$) are constants dependent upon the stress state. Equations (73) and (74) are identical in form, and therefore the determination of the instability stresses in this problem is a task exactly comparable with that of the determination of the instability stresses in the previous problem. Equation (74) is given in Ref. 2 and also, for the particular case σ_2 equal to zero, in Ref. 3.

In the discussion of the assumption that plastic deformation only occurs, the analysis proceeds differently according as $K' > 0$ (Case A) or $K' < 0$ (Case B).

Case A. From Eq. (65) it follows that φ' satisfies the equation

$$\nabla_1^4 \varphi' = \beta\{\delta_1 r_1^{-1} w'_{.\xi\xi\xi\xi} - (\delta_1 + \delta_2) w'_{.\xi\xi\eta\eta} + \delta_2 r_2^{-1} w'_{.\eta\eta\eta\eta}\} \tag{75}$$

and satisfies boundary conditions necessarily consistent with the assumption that loading occurs everywhere. If this assumption is true, then at every point of the middle surface, it is untrue that $-1 \leq \zeta^*(P) \leq 1$. The critical case occurs when $\zeta^*(P) = -1$. Now suppose that, in general, at some particular point of the middle surface $\zeta^*(P) =$

$-(1 + \delta)$; for simplicity, it is assumed that δ is real. Then, from Eq. (45), it is straightforward to prove that

$$N\delta^2 - \delta + \delta_1 = 0 \tag{76}$$

where

$$\delta_1 = \{(\sigma_1 - 2\sigma_2)\varphi'_{,\xi\xi} + (\sigma_2 - 2\sigma_1)\varphi'_{,\eta\eta}\} \Big/ \left(\beta \frac{K'}{L}\right) - 1 \tag{77}$$

and hence $\delta \geq \delta_1$ as $N \geq 0$. The critical case will occur if $\delta = 0$ and then $\delta_1 = 0$. A sufficient condition for loading to occur over the entire thickness of the plate is that $\delta_1 \geq 0$, i.e. from Eq. (77) that

$$\{(\sigma_1 - 2\sigma_2)\varphi'_{,\xi\xi} + (\sigma_2 - 2\sigma_1)\varphi'_{,\eta\eta}\} \Big/ \left(\beta \frac{K'}{L}\right) \geq 1. \tag{78}$$

Case B. It is straightforward to show that the analysis of Case *B* is deducible from the analysis of Case *A*. Full details are given in Ref. 4, and here only the results are stated. The relation corresponding to relation (78) is

$$\{(\sigma_1 - 2\sigma_2)\varphi'_{,\xi\xi} + (\sigma_2 - 2\sigma_1)\varphi'_{,\eta\eta}\} \Big/ \left(-\beta \frac{K'}{L}\right) \geq 1. \tag{79}$$

Therefore a sufficient condition, applicable in either case, for loading to occur over the entire thickness of the plate is that

$$\{(\sigma_1 - 2\sigma_2)\varphi'_{,\xi\xi} + (\sigma_2 - 2\sigma_1)\varphi'_{,\eta\eta}\} \Big/ \left(\frac{\beta}{L} \mid K' \mid\right) \geq 1. \tag{80}$$

Here it is required that loading occurs over the whole of the plate. Therefore it sufficient that φ' is determined from Eq. (75) subject to boundary conditions consistent with the condition (80) for loading everywhere. The accurate determination of φ' may not always be very simple. However, the following approximate analysis indicates that it should always be possible to choose the rate-of-change of applied edge stress at instability so that loading occurs over the entire plate, although the determination of the least possible rates of change is, in general, a difficult problem. Now if the instability stresses do not greatly exceed the elastic limit, then it is not unreasonable to approximate Eq. (75) by the equation

$$\nabla_1^4 \varphi' = 0. \tag{81}$$

Next take the boundary conditions on φ' to be that (i) $\varphi'_{,ss}$ is constant along any edge and has the same value for opposite edges and (ii) $\varphi'_{,ns}$ is zero along all edges. The solution of Eq. (81) subject to these boundary conditions corresponds to a uniform stress-rate distribution. The condition (80) is now simply

$$\text{const.} \Big/ \frac{\beta}{L} K' \geq 1, \tag{82}$$

and loading will occur over the entire plate thickness if $\sigma_1' = -\varphi'_{,\eta\eta}$ and $\sigma_2' = -\varphi'_{,\xi\xi}$ are such that

$$(2\sigma_1 - \sigma_2)\sigma_1' + (2\sigma_2 - \sigma_1)\sigma_2' \geq \frac{\beta}{L} \text{Max.} \mid K' \mid. \tag{83}$$

The above analysis of Problems A and B refers to two salient problems of the general analysis; the first corresponds to greatest possible instability stresses and the second corresponds to least possible instability stresses; and, between these extremes, it appears probable that there exists a continuous set of instability stresses, this corresponding to a continuous set of changes of the applied edge stresses.

In Refs. 1, 2 and 3 some examples of Problems A, A and B, and B, respectively are considered, but in all cases there is implicit neglect of strain compatibility. In Ref. 4, a detailed analysis is given of two cases, viz. (i) an infinitely-long simply-supported plate under uniform compression stress along one pair of edges and (ii) a finite simply-supported plate under uniform compression stress along one pair of opposite edges. Problems A and B are solved first accurately and then approximately (through neglect of strain compatibility) in case (i) and are solved approximately in case (ii). The approximate solutions may be expected to approximate closely the accurate solutions when the instability stresses are not too greatly in excess of the elastic limit.

5. Conclusions. There are two salient problems, often designated in the published literature as the von Kármán and Shanley problems; in the first, which corresponds to greatest possible instability stresses, it is postulated that the changes in the applied edge stresses at instability are zero; and in the second, which corresponds to least possible instability stresses, it is postulated that the changes in the applied edge stresses at instability are such that plastic deformation only occurs. It appears probable that, between these extremes, there exists a continuous set of instability stresses, this corresponding to a continuous set of changes in the applied edge stresses.

In the first problem, if strain compatibility is neglected, then the determination of the instability stresses is straightforward; it is probable that these approximate values underestimate the accurate values on the basis of the present theory by an amount that is small only if the instability stresses do not greatly exceed the elastic limit. In the second problem, the determination of the instability stresses is straightforward; however, although the assumption that only plastic deformation occurs at instability is probably correct, the determination of the corresponding least possible changes in the applied edge stresses is in general difficult.

It is known that theorectical values, based on the present flow theory, considerably overestimate experimental values of plastic buckling loads for long simply-supported plates under end compression. Further, deformation theories are available that provide results in good agreement with the experimental results; however, this fact is sometimes taken as a justification for a deformation, as opposed to a flow, theory of plasticity, an such a justification is untenable. If the experimental results relate to perfect test specimens perfectly loaded or, if these results are not particularly sensitive to imperfections of geometry, loading or mechanical behaviour, then the disagreement should be traceable to the assumed mechanical behaviour; otherwise, the disagreement should be traced to either imperfections or assumed mechanical behaviour or both of these matters.

Acknowledgement

The author is indebted to Professors W. Prager and D. C. Drucker of Brown University and to Professor G. H. Handelman of Carnegie Institute of Technology for comments on the analysis presented in this paper.

References

1. G. H. Handelman and W. Prager, Nat. Adv. Comm. Aero., Washington, *Tech. Note No. 1530*, 1948.
2. H. G. Hopkins, RESEARCH Engineering Structures Supplement: Vol. II of the Colston Papers (London, 1949), 93.
3. C. E. Pearson, Jour. Aero. Sci., **17**, 417, (1950).
4. H. G. Hopkins, Graduate Division of Applied Mathematics, Brown University, *ONR Report A11-67*, December 1951.
5. W. Prager, Jour. Applied Phys., **20**, 235, (1949).
6. D. C. Drucker, (a) Quart. Applied Math., **7**, 411, (1950); (b) Graduate Division of Applied Mathematics, Brown University, *ONR Report A11-58*, May 1951. To appear in Proc. 1st, U.S. Nat. Cong. Applied Math.
7. S. Timoshenko, *Theory of elastic stability* (New York and London, 1936).

WATER WAVES OVER A CHANNEL OF INFINITE DEPTH†

By

THOM R. GREENE (*The General Electric Co., Schenectady, N. Y.*)

AND ALBERT E. HEINS (*Carnegie Institute of Technology*)

I. Introduction. This is a continuation of some work on gravity waves in channels of finite depth with special types of obstacles (1), (2), (3). Thus far three cases have been considered: the channel of finite depth with (i) a dock; (ii) a submerged plane barrier and (iii) the joining of two free surfaces. In this paper and one to follow we propose to discuss the corresponding problems for channels of infinite depth. The problem which concerns us here is case (ii) and its limiting case (i) for such channels.

The physical background for the problem has been discussed in (1) and (4). Here we shall reformulate the problem so that the notion of infinite depth is admitted. As we have done in the past, only propagating regions will be considered here, the possibility of non-propagating regions may be treated in a fashion similar to that of the propagating region. It will be assumed that the waves have propagation normals which are not perpendicular to the edge of the barrier and in the beginning we shall assume that the barrier is submerged (2). As limiting cases, we will get the case of normal incidence and the "dock problem."

While there is a certain parallelism between the formulation of the problem in channels of infinite and finite depth, we observe that the mathematical methods of solution become more subtle in the channels of infinite depth. Still, we provide here methods which will solve the problem we have described. With minor modifications in method but definite changes in formulation, we can solve (iii) for infinite channels.

II. Fundamental Equations of Motion and Statement of Problem. If we use the linearized equations of motion as we have done in the past, we have for an incompressible, non-viscous fluid, in irrotational motion that

$$\phi'_{xx} + \phi'_{yy} + \phi'_{zz} = 0$$

where ϕ' is the velocity potential. On a rigid surface $\phi'_n = 0$, while on a "free surface"

$$g\phi'_n + \phi'_{tt} = 0.$$

Assuming monchromatic time dependence

$$\phi' = \phi(x, y, z) \exp(-i\omega t)$$

and these conditions become $\phi_n = 0$ and $g\phi_n = \omega^2\phi$ respectively.

Here we are concerned with a channel of infinite depth with a semi-infinite plane barrier submerged a units below the undisturbed free surface (See Figure 1). We shall assume that wave motion exists for $|x| \to \infty$, $y = 0$ where the propagation normal is parallel to the free surface but not at right angles to the edge of the barrier, that is the z axis. As such there will be a z variation $\exp(ikz)$ $(k > 0)$ in ϕ which we may suppress

†Received May 22, 1952. This paper is based on research conducted in part under a contract between the Office of Ordnance Research of the Department of the Army and Carnegie Institute of Technology. The paper was submitted by T. R. Greene in partial fulfillment of the requirements for the degree of Doctor of Science at the Carnegie Institute of Technology.

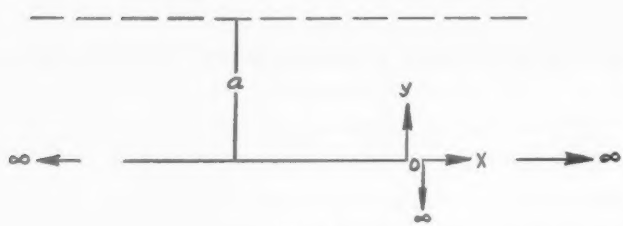

FIG. 1.

from the beginning. This gives us as fundamental equations of motion

$$\phi_{xx} + \phi_{yy} - k^2\phi = 0 \qquad \text{in the fluid medium,} \tag{2.1}$$

$$\phi_n = 0 \qquad \text{on a rigid surface, and} \tag{2.1'}$$

$$\phi_y = \beta\phi \qquad \beta = \omega^2/g \qquad \text{on a free surface.} \tag{2.1''}$$

Equation (2.1) possesses the following types of bounded solutions. For $x \to \infty$, $y < a$

$$\phi = \exp\,[ikz + \beta y \pm i\{\beta^2 - k^2\}^{1/2}x]$$

while for $x \to -\infty$, $0 \leq y \leq a$

$$\exp\,(ikz \pm i\kappa x)\cosh \alpha_0 y/a$$

where

$$\kappa = \{(\alpha_0/a)^2 - k^2\}^{1/2}$$

and

$$\alpha \sinh \alpha = \beta a \cosh \alpha \tag{2.2}$$

has the smallest real roots $\pm\, \alpha_0$. Eq. (2.2) also has an infinite sequence of imaginary roots $i\alpha_n$, $\alpha_{-n} = -\alpha_n$, $n = 1, 2, \cdots$. Wave motion exists if $\beta > k$ and $|\, \alpha_0/a \,| > k$.

III. Derivation of the Integral Equation. Following methods which have been discussed elsewhere (1), (2), (3), it is a fairly straightforward task to formulate the integral equation which describes the boundary value problem considered in Sec. II. In view of the fact that we have suppressed the z dependence in ϕ, we have essentially a two dimensional problem. We shall divide the half space $y \leq a$ into two regions: (a) the strip $0 \leq y \leq a$, $-\infty < x < \infty$ which we shall henceforth refer to as R_1 and (b) the half plane $y \leq 0$, $-\infty < x < \infty$ which we shall refer to as R_2 . Since we require the use of the Helmholtz theorem in its improper form to provide representations of $\phi(x, y)$ in R_1 and R_2 in terms of $\phi_y(x, 0)$, some statements regarding the growth of ϕ as $r \to \infty$ will be in order ($r = [x^2 + y^2]^{1/2}$). To begin with, in R_1 for $x \to \infty$, $\phi(x, y)$ is asymptotic to

$$[\gamma_1 \exp\,(i\kappa x) + \gamma_2 \exp\,(-i\kappa x)] \cosh \alpha_0 y/a. \tag{3.1}$$

This is tantamount to the statement that the asymptotic form of ϕ for $x \to -\infty$ is the bounded solution which would have been obtained had the barrier extended to infinity on the right. For $x \to \infty$ and $y \leq a$, we specify the asymptotic form of ϕ as

$$[\gamma_3 \exp\,(i\sigma x) + \gamma_4 \exp\,(-i\sigma x)] \exp\,(\beta y) \tag{3.2}$$

where $\sigma = \{\beta^2 - k^2\}^{1/2}$ and the positive root is to be understood. This again is a statement that for a channel of infinite depth without a barrier (3.2) is the bounded solution. One final task remains and that is to specify the nature of ϕ as $r \to \infty$, $x < 0$ and $y < 0$. In this we are not as fortunate as we were in the other two cases. We shall assume (and verify with the solution of the problem) that $\phi = 0\,[\exp(k_1 r)]$ for $y < 0$, $x < 0$, $r \to \infty$ where $k_1 < k$. This condition will enable us to write a representation for ϕ in R_2, that is for $y \leq 0$. One final assumption is required and that is the statement that ϕ and ϕ_y are integrable for all finite x — again subject to final verification.

If we introduce a Green's function $G^{(1)}(x, y, x', y')$ associated with Eq. (2.1) and which satisfies the boundary conditions

$$G_y^{(1)} = \beta G^{(1)} \qquad y = a, \qquad -\infty < x < \infty \tag{3.3}$$

and

$$G_y^{(1)} = 0 \qquad y = 0, \qquad -\infty < x < \infty \tag{3.3$'$}$$

and the further conditions at infinity that $G^{(1)} \to 0$ for $x' - x \to \infty$ while for $x - x' \to \infty$

$$G^{(1)} = \frac{4\alpha_0 \cosh \alpha_0 y/a \cosh \alpha_0 y'/a \sin \kappa(x - x')}{a\kappa(2\alpha_0 + \sinh 2\alpha_0)}$$

we have as a representation for ϕ in R_1

$$\phi(x, y) = [\cosh \alpha_0 y/a][\gamma_1 \exp(i\kappa x) + \gamma_2 \exp(-i\kappa x)] - \int_0^\infty G^{(1)}(x, y, x', 0)\phi_{y'}(x', 0)\, dx'. \tag{3.4}$$

Here (3)

$$G^{(1)}(x, y, x', y') = \sum_{n=1}^{\infty} \frac{2\alpha_n(\cos \alpha_n y/a)(\cos \alpha_n y'/a) \exp[-|x - x'|\{k^2 + \alpha_n^2/a^2\}^{1/2}]}{a(2\alpha_n + \sin 2\alpha_n)\{\alpha_n^2/a^2 + k^2\}^{1/2}}$$

$$+\,0 \quad \text{if} \quad x' > x \quad \text{or}$$

$$+\,\frac{4\alpha_0(\cosh \alpha_0 y/a)(\cosh \alpha_0 y'/a) \sin \kappa(x' - x)}{a\kappa(2\alpha_0 + \sinh 2\alpha_0)} \qquad \text{if} \qquad x' < x.$$

For the region R_2 we have in a similar fashion

$$\phi(x, y) = \int_0^\infty G^{(2)}(x, y, x', 0)\phi_{y'}(x', 0)\, dx' \tag{3.5}$$

where

$$G^{(2)}(x, y, x', y') = \frac{1}{2\pi}\,[K_0(k\{(x - x')^2 + (y - y')^2\}^{1/2}) + K_0(k\{(x - x')^2 + (y + y')^2\}^{1/2})].$$

The function $K_0(\delta)$ is the solution of the differential equation

$$y_{\delta\delta} + y_\delta/\delta - y = 0$$

which is $0(\log \delta)$, $\delta \to 0^+$. It is a Bessel function of the second kind which possesses the property that it is of the order $\exp(-\delta)/\delta^{1/2}$ for $\delta \to \infty$. This last order condition accounts for the choice of the constant k_1. On the basis of these two representations of

$\phi(x, y)$ in R_1 and R_2 we can form an integral equation of the Wiener-Hopf type. We note that $\phi(x, y)$ is continuous at $y = 0$ for all $x > 0$. We have then for $x > 0$

$$\gamma_1 \exp(i\kappa x) + \gamma_2 \exp(-i\kappa x) - \int_0^\infty [G^{(1)}(x, 0, x', 0) + G^{(2)}(x, 0, x', 0)]\phi_{y'}(x', 0)\, dx' = 0. \tag{3.6}$$

IV. The Fourier Transform of the Integral Equation. We now define Eq. (3.6) to read for all x

$$I(x) = J(x) - \int_{-\infty}^\infty [G^{(1)}(x, 0, x', 0) + G^{(2)}(x, 0, x', 0)]L(x')\, dx'. \tag{4.1}$$

Here

$$\begin{aligned} I(x) &\equiv 0 \qquad \text{for} \qquad x > 0 \\ J(x) &\equiv \gamma_1 \exp(i\kappa x) + \gamma_2 \exp(-i\kappa x) && x > 0 \\ &\equiv 0 && x < 0 \\ L(x) &\equiv 0 && x < 0 \\ &\equiv \phi_{y'}(x) && x > 0 \end{aligned}$$

Clearly then $I(x) = 0\,[\exp(kx)]$ for $x \to -\infty$ and $L(x) = 0\,[\exp k_1 x]$ for $x \to \infty$. Actually $L(x)$ is bounded for $x \to \infty$, being of the form $\exp(\pm i\sigma x)$ but this last description of $\phi(x, 0)$ for $x \to \infty$ need not be stated explicitly.

Let us now introduce the Fourier transforms of $I(x)$, $J(x)$ and $L(x)$ as well as those of $G^{(1)}(x, 0, 0, 0)$ and $G^{(2)}(x, 0, 0, 0)$. Henceforth we shall use the notation $I(w)$ to denote the Fourier transform of $I(x)$, $J(w)$ for $J(x)$, etc. We have from Eq. (4.1)

$$I(w) = \frac{\gamma_1}{i(w - \kappa)} + \frac{\gamma_2}{i(w + \kappa)} - L(w)[G^{(1)}(w) + G^{(2)}(w)] \tag{4.2}$$

where

$$G^{(1)}(w) = \int_{-\infty}^\infty \exp(-iwx)G^{(1)}(x, y, 0, y')\, dx$$

$$= \frac{\cosh \rho s[\beta \sinh \rho(t - a) + \rho \cosh \rho(t - a)]}{\rho[\rho \sinh \rho a - \beta \cosh \rho a]}$$

and $(s, t) = (y, y')$ if $y < y'$ or (y', y) if $y > y'$. Furthermore

$$G^{(2)}(w) = \int_{-\infty}^\infty \exp(-iwx)G^{(2)}(x, y, 0, y')\, dx$$

$$= \{\exp[-\rho \mid y - y' \mid] + \exp[-\rho \mid y + y' \mid]\}/2\rho.$$

where $\rho = \{k^2 + w^2\}^{1/2}$. The branch of ρ has been chosen which reduces to k when $w = 0$.

We now examine the regions of regularity of the various transforms. First $I(w)$ is

regular in the upper half plane $v > -k$ $(w = u + iv)$, $J(w)$ is regular in the lower half plane $v < 0$, and $L(w)$ in the lower half plane $v < -k_1$. This follows from the assumed integrability for finite x of $I(x)$, $L(x)$ and $J(x)$ as well as their growths for $|x| \to \infty$. Similarly $G^{(1)}(w)$ is regular in the strip $-\{k^2 + \alpha_1^2/a^2\}^{1/2} < v < 0$ while $G^{(2)}(w)$ is regular in the strip $-k < v < k$. There is then a common strip of analyticity for all of these transforms, the application of the Fourier transform to Eq. (3.4) is permissible and we obtain Eq. (4.2). Upon simplifying we get

$$I(w) = \frac{\gamma_1}{i(w - \kappa)} + \frac{\gamma_2}{i(w + \kappa)} - K(w)L(w) \tag{4.3}$$

where

$$K(w) = [1 - \beta/\rho][\exp a\rho]/[\rho \sinh \rho a - \beta \cosh \rho a].$$

Following Wiener and Hopf (5), Eq. (4.3) may be rewritten

$$K_+(w)I(w) - \gamma_1 \frac{K_+(w) - K_+(\kappa)}{i(w - \kappa)} - \gamma_2 \frac{K_+(w) - K(-\kappa)}{i(w + \kappa)}$$

$$= \frac{\gamma_1 K_+(\kappa)}{i(w - \kappa)} + \frac{\gamma_2 K(-\kappa)}{i(w + \kappa)} - L(w)K_-(w) \tag{4.4}$$

where $K_+(w)$ is regular in the upper half plane $v > -k$ while $K_-(w)$ is regular in the lower half plane $v < 0$. Eq. (4.4), as a result of an argument of analytic continuation described by Wiener and Hopf, implies that each side if individually an entire function. Our next goal is to determine the explicit forms of $K_-(w)$ and $K_+(w)$ as well as the entire function of separation.

V. The Determination of $K_-(w)$ and $K_+(w)$. Thus far we have pursued a familiar course to arrive at Eq. (4.4). From this point on we encounter a number of unfamiliar situations. In the decomposition of $K(w)$ we have three individual components to consider:

(a) $\rho \sinh \rho a - \beta \cosh \rho a$

(b) $\exp(\rho a)$

(c) $1 - \beta/\rho$

Component (a) is familiar and has been discussed in (2). The type of factoring we employ implies that the portion of (a) which is regular in some lower half plane also has a regular reciprocal in the same half plane. The same remark may be addressed to the upper half plane factor. We have then

$$\rho \sinh \rho a - \beta \cosh \rho a = M_+(w)/M_-(w)$$

where $M_+(w)$ is regular in the upper half plane $v > -\alpha_1\{1 + a^2k^2/\alpha_1^2\}^{1/2}$ and $M_-(w)$ is regular in the lower half plane $v < 0$. Here

$$M_+(w) = \prod_{m=1}^{\infty} [\{1 + a^2k^2/\alpha_m^2\}^{1/2} - iaw/\alpha_m] \exp(iaw/m\pi)$$

and

$$1/M_-(w) = \prod_{m=1}^{\infty} [\{1 + a^2k^2/\alpha_m^2\}^{1/2} + iaw/\alpha_m] \exp(-iaw/m\pi) \frac{a^2\beta}{\alpha_0^2}(w^2 - \kappa^2).$$

For

$$|w| \to \infty \qquad Imw > -\{\alpha_1^2 + a^2k^2\}^{1/2}$$

$$M_+(w) = 0[w\Gamma(-iaw/\pi)\exp(-\gamma iaw/\pi)]^{-1}$$

where γ is the Euler-Mascheroni constant, while for $|w| \to \infty$, $Imw < 0$

$$M_-(w) = 0[\exp(+iaw\gamma/\pi)\Gamma(iaw/\pi)w^{-1}].$$

Now we turn to (b). The decomposition of $\exp(\rho a)$ multiplicatively is equivalent to the decomposition of additively. $\rho = \{w^2 + k^2\}^{1/2}$ is regular in the entire w plane provided cuts are introduced along the imaginary axis from k to infinity and $-k$ to negative infinity. We define

$$\{w^2 + k^2\}^{1/2} = |w^2 + k^2|^{1/2}[\exp\{i/2\arg(w - ik) + i/2\arg(w + ik)\}]$$

where

$$-3\pi/2 < \arg(w - ik) < \pi/2$$

$$-\pi/2 < \arg(w + ik) < 3\pi/2.$$

With these definitions at hand we turn to the decomposition of $1/\rho$ from which we can readily obtain that of ρ. From Cauchy's theorem we have

$$2\pi i/(w^2 + k^2)^{1/2} \equiv \int_C dz/(z - w)(z^2 + k^2)^{1/2} \tag{5.1}$$

where C is a closed rectangular path in the strip $-k < Im\ z < k$ whose vertical contributions disappear as they recede to infinity. The upper side of the rectangle may be deformed into a path along the branch cut from k to infinity, while the lower one goes into a path over the lower cut and we get

$$\pi i/(w^2 + k^2)^{1/2} = \int_k^\infty dy/(w - iy)(y^2 - k^2)^{1/2} - \int_k^\infty dy/(w + iy)(y^2 - k^2)^{1/2}.$$

This provides a separation of the required type—the first term being regular in the lower half plane $v < k$, the second in the upper half plane $v > -k$. For our purposes, however, we require more explicit information, particularly as to the nature of these integrals as $|w| \to \infty$ and we therefore proceed with the evaluation of these integrals.

As a matter of notation we write

$$\pi i(w^2 + k^2)^{-1/2} = f(w) + f(-w)$$

where

$$f(w) = \int_k^\infty dy/(w - iy)(y^2 - k^2)^{1/2} = (w^2 + k^2)^{-1/2}\log[(1 + t)/(1 - t)] \tag{5.2}$$

and

$$t = (w + ik)^{1/2}(w - ik)^{-1/2}. \tag{5.3}$$

Upon using the definitions of the $\arg(w + ik)$ and $\arg(w - ik)$ which we wrote above,

we find that the cut w plane (Figure 2) maps into the upper half of the t plane by means of (5.3). Further the transformation $z = (1 + t)/(1 - t)$ is bilinear and hence maps the upper half plane $Imt > 0$ into the upper half plane $Imz > 0$ (Figure 2). We therefore define the branch of the logarithm by $0 \leq \arg z \leq \pi$. Given this information, we may verify that $f(w)$ is regular in the lower half plane $v < k$. The factor $(w^2 + k^2)^{-1/2}$ multiplying the logarithm in $f(w)$ changes sign as we cross the lower branch cut. But the

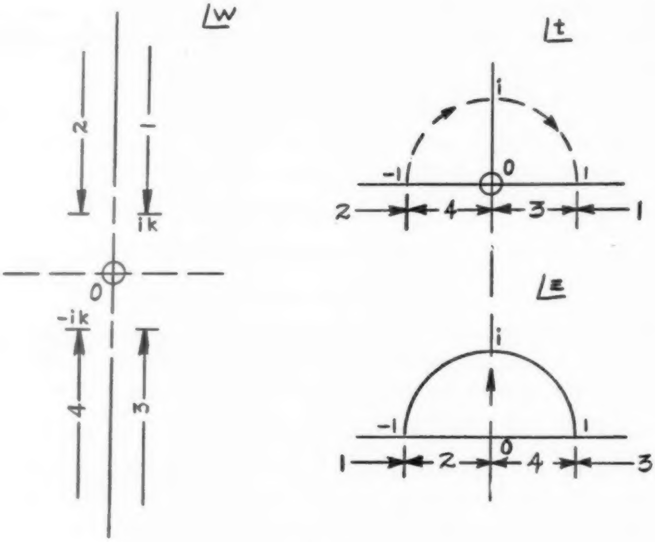

FIG. 2.

$\arg z = 0$ on this cut so that the real part of $f(w)$ is continuous. In crossing the branch cut $\log | z |$ becomes $- \log | z |$ so that the imaginary part of $f(w)$ is continuous. Hence $f(w)$ is regular in the required lower half plane. On the upper branch out the $\arg z = \pi$ and hence the sign of the radical multiplier introduces a discontinuity.

Similar comments apply to the function $f(-w)$. The transformation $t' = (w - ik)^{1/2} \cdot (w + ik)^{-1/2}$ sends the cut plane into the half plane $Imt' < 0$ and the transformation $z' = (1 - t')(1 + t')^{-1}$ takes the half plane $Imt' < 0$ into the upper half plane $Imz' > 0$ with the upper branch cut mapped into the positive z' axis. We put $0 \leq \arg z' \leq \pi$ to define the branch of the logarithm in $f(-w)$, that is

$$f(-w) = (w^2 + k^2)^{-1/2} \log (1 - t)(1 + t)^{-1}$$

and we find that $f(-w)$ is now regular in the upper half plane $v > -k$.

As a check, we have in the strip $-k < v < k$

$$f(w) + f(-w) = (w^2 + k^2)^{-1/2}[\log z + \log z']$$

$$= (w^2 + k^2)^{-1/2}[\log | zz' | + i \arg z + i \arg z'].$$

Since $z = -1/z'$, this last expression reduces to the one with which we started. Hence if we write $\exp [a(w^2 + k^2)^{1/2}]$ in product form as we have already described, we find that

$$\exp [a(u^2 + k^2)^{1/2}] = N_-/N_+$$

where

$$N_-(w) = \exp [a(w^2 + k^2)f(w)/\pi i]$$

$$1/N_+(w) = \exp [a(w^2 + k^2)f(-w)/\pi i].$$

The asymptotic forms of $f(w)$ and $f(-w)$ will be required at a later point in our work and since the analytical machinery is now developed, we shall provide these forms. Note first that

$$\log (1 + t)(1 - t)^{-1} = \log \{i[w + (w^2 + k^2)^{1/2}]/k\}.$$

Further for $u > 0$, $v < -k$, $-\pi/2 < \arg(w^2 + k^2)^{1/2} < 0$ and $-\pi/2 < \arg w < 0$. Hence for $|w| \to \infty$, $v < -k$, $u > 0$,

$$\log \{i[w + (w^2 + k^2)^{1/2}]/k\} = \log [(2iw/k) + (ik/2w)]$$

up to terms of $0(1/w)$. This in turn has the expansion

$$(\log 2iw/k) - k^2/4w^2. \tag{5.4}$$

Now the choice of the definition of the logarithm is dictated by the fact that the argument of the logarithm in (5.4) lies between the limits 0 and π. Hence if we consider the principal branch of the logarithm of w we see that $\log i = i\,\pi/2$ and hence

$$\log (1 + t)(1 - t)^{-1} = i\pi/2 + \log 2/k + \log w + 0(1/w^2) \tag{5.4'}$$

in this quadrant. If we now turn to the third quadrant, we see that we get certain minor changes but the same dominant term. In this quadrant $0 < \arg(w^2 + k^2)^{1/2} < \pi/2$ while $-\pi < \arg w < -\pi/2$. Then

$$\log (1 + t)(1 - t)^{-1} = \log [i\{w + (w^2 + k^2)^{1/2}\}/k]$$

$$= -\log 2iw/k + 0(1/w^2)$$

for $u < 0$, $v < -k$, $|w| \to \infty$. But again the logarithmic term has an argument between 0 and π and since the arg w lies between $-\pi$ and $-\pi/2$, the negative of this argument lies between $\pi/2$ and π. Hence as we might have surmised

$$\log (1 + t)(1 - t)^{-1} = -i\pi/2 - \log w - \log 2/k + 0(1/w^2)$$

for $|w| \to \infty$, $v < +k$, $u < 0$. This change in the asymptotic form of the logarithm of $\log(1 + t)(1 - t)^{-1}$ in the third quadrant does not change the dominant term in the asymptotic form of $f(w)$ however and (5.4′) is still in force. Similarly

$$f(-w) = -[\log w + \log 2/k - i\pi/2 + 0(1/w^2)]/w$$

for $|w| \to \infty$, $v > -k$. From this we conclude that

$$N_-(w) = \exp [aw/2 + (aw/\pi i)(\log 2/k) + (aw/\pi i) \log w]$$

if we retain the non-vanishing terms for $|w| \to \infty$, $v < +k$, while

$$1/N_+(w) = \exp [(-aw/2) - (aw/\pi i)(\log 2/k) - (aw/\pi i)(\log w)]$$

for $|w| \to \infty$, $v > -k$.

The factoring of $1 - \beta/(w^2 + k^2)^{1/2}$ into the product of two terms, one of which is regular in the lower half plane $v < 0$, while the other of which is regular in the upper half plane $v > -k$ is guaranteed by the original work of Wiener and Hopf. However, rather than apply their original methods, we shall follow an approach which will enable us to express the factors in terms of $f(w)$ and $f(-w)$. Again following the lead from the form (b), we write

$$1 - \beta/(w^2 + k^2)^{1/2} = \exp\,[\ln\,\{1 - \beta/(w^2 + k^2)^{1/2}\}]$$

where we choose the branch of $(w^2 + k^2)^{1/2}$ which reduces to k for $w = 0$ as well as the principal branch of the logarithm. Instead of decomposing the logarithmic factor additively, we shall decompose its derivative in this fashion since differentiation will not alter the open strip of regularity. Hence we shall examine first

$$\frac{d}{dw}\ln\,[1 - \beta/(w^2 + k^2)^{1/2}] = w\beta/(w^2 + k^2)[\{w^2 + k^2\}^{1/2} - \beta]$$

$$= -\frac{1}{2(w - ik)} - \frac{1}{2(w + ik)} + \frac{w}{w^2 - \sigma^2} - \frac{w}{\beta\{w^2 + k^2\}^{1/2}} + \frac{w\{w^2 + k^2\}^{1/2}}{\beta\{w^2 - \sigma^2\}}. \tag{5.5}$$

Now of these five terms, we can state the following. The first and third are regular in the lower half plane $v < 0$, while the second is regular in the upper half plane $v > -k$. The fourth and fifth terms are regular only in the strips $-k < v < k$ and $-k < v < 0$ respectively. The fourth term however is the derivative of $-(w^2 + k^2)^{1/2}/\beta$ which has been discussed in (b). The fifth term may be decomposed as follows:

$$\frac{w\{w^2 + k^2\}^{1/2}}{\beta\{w^2 - \sigma^2\}} = \frac{\{w^2 + k^2\}^{1/2}}{2\beta}\left\{\frac{1}{w + \sigma} + \frac{1}{w - \sigma}\right\}.$$

We know how to decompose $(w^2 + k^2)^{1/2}$ additively so that this last term breaks up into four terms, two of which are regular in the lower half plane $v < 0$, while the other two are still regular in the strip $-k < v < 0$. But the simple poles in these last terms may be extracted quite readily so that we finally have that the portion of (5.5) which is regular in the appropriate lower half plane is

$$\begin{aligned} U_-(w) = &-\frac{1}{2(w - ik)} + \frac{w}{w^2 - \sigma^2} - \frac{1}{\pi\beta i}\frac{d}{dw}(w^2 + k^2)f(w) \\ &+ \frac{(w^2 + k^2)f(w)}{2\beta\pi i(w + \sigma)} + \frac{(w^2 + k^2)f(w)}{2\beta\pi i(w - \sigma)} \\ &+ \frac{\beta f(+\sigma)}{2\pi i(w + \sigma)} + \frac{\beta f(-\sigma)}{2\pi i(w - \sigma)}. \end{aligned}$$

On the other hand, the term

$$\begin{aligned} U_+(w) = &-\frac{1}{2(w + ik)} - \frac{1}{\pi\beta i}\frac{d}{dw}(w^2 + k^2)f(w) \\ &+ \frac{1}{2\beta\pi i(w + \sigma)}[(w^2 + k^2)f(-w) - \beta^2 f(+\sigma)] \\ &+ \frac{1}{2\beta\pi i(w - \sigma)}[(w^2 + k^2)f(-w) - \beta^2 f(-\sigma)] \end{aligned}$$

is regular in the appropriate upper half plane. Hence

$$1 - \beta/\rho = \left[\exp \int^{w} U_-(w)\, dw\right]\left[\exp \int^{w} U_+(w)\, dw\right]$$

where the first factor is regular in the appropriate lower half plane while the second is regular in the appropriate upper half plane. The constants of integration are chosen so that the left side matches the right side in the strip of regularity.

The asymptotic forms of $\exp \int^w U_-(w)dw$ and $\exp \int^w U_+(w)dw$ are readily determined from our results in (*b*). We have for $|w| \to \infty,\ v < 0$

$$U_-(w) = 1/w + 0(1/w^2)$$

while

$$U_+(w) = -1/w + 0(1/w^2)$$

for $|w| \to \infty,\ v > -k$. Hence

$$\int^{w} \exp U_-(w)\, dw = cw$$

and

$$\int^{w} \exp U_+(w)\, dw = c_1/w$$

in these respective cases where the constants c_1 and c satisfy the relation $cc_1 = 1$. The precise values of the constants do not concern us here.

Now we may put together the factors (*a*), (*b*) and (*c*) to determine $K_-(w)$. In order to obtain an entire function of separation which is of algebraic growth, we introduce into $K_-(w)$ a factor $\exp \chi(w)$ where $\chi(w)$ is to be chosen to make $K_-(w)$ of algebraic order for $v < k_1$, $|w| \to \infty$. This process has already been described elsewhere (1). We have then, for $v < k_1$, $|w| \to \infty$

$$K_-(w) = 0(w^{-1/2}) \exp [(iaw/\pi)(\gamma - 1 + \ln ak/2\pi) + \chi(w)].$$

From this we see that if $\chi(w)$ is chosen as

$$\chi(w) = [1 - \gamma - \ln (ak/2\pi)](iaw/\pi)$$

$K_-(w) = 0(w^{-1/2})$ in this lower half plane. Upon examining $K_+(w)$ for $|w| \to \infty,\ v > -k$ we find that $K_+(w) = 0(w^{+1/2})$. Hence the factoring of $K(w)$ and the asymptotic forms of the factors are known explicitly.

In order to determine $E(w)$, the entire function of separation, we merely apply an extended form of Liouville's theorem. For example, for $|w| \to \infty,\ v < 0,\ E(w) = 0(w^{-1})$, while for $|w| \to \infty,\ v > -k,\ E(w) = 0(w^{1/2})$ as we can readily see from Eq. (4.4). From this we see that since $E(w)$ is a polynomial, $E(w)$ is identically zero.

Earlier in our work we had assumed that $\phi_\nu(x, 0)$ was integrable from $x = 0$ to any finite x. In view of the fact that $I(w) = 0(w^{-1})$ and $J(w) = 0(w^{-1/2})$ for $|w| \to \infty$, v suitable limited, we see that $I(x)$ is bounded for $x \to 0^-$, while $\phi_\nu(x, 0) = 0(x^{-1/2})$ for $x \to 0^+$, thus verifying the integrability for finite x for these functions. The precise forms of ϕ will be investigated in the next section.

VI. The Determination of $\phi(x,y)$. In order to determine $\phi(x,y)$ we return to the Helmholtz representations for ϕ in R_1 and R_2. For R_1 we have from (3.5)

$$\phi(x, y) = [\cosh \alpha_0 y/a][\gamma_1 \exp (i\kappa x) + \gamma_2 \exp (-i\kappa x)] - \frac{1}{2\pi} \int_C G^{(1)}(w, y, 0) L(w) \exp (iwx)\, dw. \qquad (6.1)$$

Here $G^{(1)}(w, y, 0)$ is the bilateral Fourier transform of $G^{(1)}(x, y, 0, 0)$ an analytic function of w in the strip $-\{k^2 + \alpha_0^2/a^2\}^{1/2} < v < 0$. The convolution is justified here since $L(w)$ is the transform of $L(x)\exp(vx)$ which is $L(-\infty, \infty)$ for $-k < v < 0$ and $0 < y < a$ (6). The path C is drawn in the strip of analyticity $-k < v < 0$ and the evaluation of the integral depends on whether $x \lessgtr 0$. If $x < 0$, we close below by a sequence of semicircular arcs passing between the poles of $G^{(1)}(w, y, 0)$ on the negative imaginary axis. In this case the integral becomes

$$-\frac{1}{2\pi}\int_C \left[\frac{\gamma_1 K_+(\kappa)}{i(w-\kappa)} + \frac{\gamma_2 K_+(-\kappa)}{i(w+\kappa)}\right]\left[\frac{\rho\cosh\rho(y-a) + \beta\sinh\rho(y-a)}{K_-(w)[\rho\sinh\rho a - \beta\cosh\rho a]}\right]\frac{\exp(iwx)\,dw}{\rho}$$

$$= \sum_{n=1}^{\infty}\left[\frac{\gamma_1 K_+(\kappa)}{(-iw_n-\kappa)} + \frac{\gamma_2 K_+(-\kappa)}{(\kappa - iw_n)}\right]\frac{\exp(w_n x)(\rho_n^2 + a^2\beta^2)\exp(w_n x)}{iK_-(-iw_n)aw_n(a\beta - a^2\beta^2 - \alpha_n^2)}$$

where $w_n = \{\alpha_n^2/a^2 + k^2\}^{1/2}$. The closing of the path C in the manner described above is permitted since the integrand is $0\,[w^{-3/2}\exp(-vx)]$ for $|w| \to \infty$, $v < 0$.

For $x > 0$, the path C in Eq. (6.1) is closed above. The singularities in the upper half plane are a branch point at $w = ik$ and four poles at $w = \pm\sigma$ and $\pm\pi$. We introduce the positive imaginary axis from ik to $i\infty$ as a branch cut. The phase of ρ has already been discussed on this cut, so the closing of C implies quarter circles on each side of the branch cut as well as integrals along the branch cut. This then supplies us with a path along which the integrand is single valued and analytic and Cauchy's theorem may be applied for the evaluation of the integral. When we do so we get the residues at the poles $\pm\sigma$ and $\pm\kappa$ and an integral along the branch cut. To do this we replace

$$K_-(w) = \frac{\exp(a\rho)(1-\beta/\rho)K_+(w)}{\rho\sinh\rho a - \beta\,\mathrm{Cosh}\,\rho a}$$

and the integral along C becomes

$$-\frac{1}{2\pi}\int_C \left[\frac{\gamma_1 K_+(\kappa)}{i(w-\kappa)} + \frac{\gamma_2 K_+(-\kappa)}{i(w+\kappa)}\right]\left[\frac{\rho\cosh\rho(y-a) + \beta\sinh\rho(y-a)}{(\exp a\rho)(\rho-\beta)K_+(w)}\right]\exp(iwx)\,dw.$$

The contributions from the quarter circles drop out as their radii becomes infinite and we are left with an integral along the cut as well as four residues. On the right side of the cut arg $= \pi/2$ while on the left side of the cut arg $\rho = -\pi/2$. Hence

$$\int_C + \int_{br.} = \sum \text{residues}$$

or

$$\int_C = -[\cosh\alpha_0 y/a][\gamma_1\exp(i\kappa x) + \gamma_2\exp(-i\kappa x)]$$

$$-\left[\frac{\gamma_1 K_+(\kappa)}{\sigma-\kappa} + \frac{\gamma_2 K_+(-\kappa)}{\sigma+\kappa}\right]\frac{\beta^2\exp\beta(y-2a)\exp(i\sigma x)}{\sigma K_+(\sigma)}$$

$$+\left[\frac{\gamma_1 K_+(\kappa)}{-(\sigma+\kappa)} + \frac{\gamma_2 K_+(-\kappa)}{\kappa-\sigma}\right]\frac{\beta^2\exp\beta(y-2a)\exp(-ix)}{\sigma K_+(\sigma)}$$

$$-\frac{i}{\pi}\int_k^{\infty}\left[\frac{\gamma_1 K_+(\kappa)}{(iv-\kappa)} + \frac{\gamma_2 K_+(-\kappa)}{(iv+\kappa)}\right]\left[\frac{T\cos T(y-a) + \beta\sin T(y-a)}{K_+(iv)(v^2+\sigma^2)}\right]$$

$$\cdot\left[\frac{\beta\cos Ta + T\sin Ta}{\exp(vx)}\right]dv$$

where $T = (v^2 - k^2)^{1/2}$.

Now we examine $\phi(x, y)$ in R_2. Here we use the representation

$$\phi(x, y) = -\int_0^\infty G^{(2)}(x, y, x', 0)\phi_{y'}(x', 0)\, dx$$

$$= -\frac{1}{2\pi}\int_C \frac{\exp(iwx)\exp(\rho y)L(w)\, dw}{\rho}$$

$$= -\frac{1}{2\pi}\int_C \frac{dw \exp(iwx + \rho y)}{\rho K_-(w)}\left[\frac{\gamma_1 K_+(\kappa)}{(w - \kappa)} + \frac{\gamma_2 K_+(-\kappa)}{(w + \kappa)}\right]$$

and again we distinguish the two cases, that is $x \gtrless 0$. For $x < 0$, the only singularity in the lower half plane is the branch point at $w = -ik$. Hence we introduce a branch cut along the imaginary axis from $-ik$ to $-i\infty$ and choose a closing path of the variety we have just described for $0 \leq y \leq a$, $x > 0$. Upon doing this we find that for $y < 0$, $x < 0$

$$\phi(x, y) = \frac{i}{\pi}\int_k^\infty \frac{\exp(vx)}{TK_-(-iv)}\left[\frac{\gamma_1 K_+(\kappa)}{(-iv - \kappa)} + \frac{\gamma_2 K_+(-\kappa)}{(-iv + \kappa)}\right]\cos Ty dv.$$

For $y < 0$, $x > 0$, we close above in a fashion similar to the case $x < 0$, $y < 0$ to obtain a function $\phi(x, y)$ which has the same functional form as we did for the region $x > 0$, $0 < y < a$. This incidently establishes the continuity of $\phi(x, y)$ for $x > 0$, $y = 0$.

VII. Remarks. We have now formulae for the region $x < 0$, $0 < y < a$, for $x > 0$, and for $x < 0$, $y < 0$. For each of these expressions the integral or infinite sum, as the case may be, is uniformly and absolutely convergent. Due to the presence of the favoring exponential when $x \neq 0$, we may carry differentiation under the integral sign (or inside the summation). It is possible, therefore, to verify directly that the Eq. (2.1) and the boundary conditions (2.1′), (2.1″) appropriate to the region are satisfied. For $x < 0$, $0 < y < a$, we note that the expansion coefficients in the series are $0(n^{-3/2})$ for $n \to \infty$. In view of the fact that we have decaying exponentials in this sum, it is clear that the asymptotic requirements of the region are met.

Consider the integral in the representation for ϕ when $x > 0$. We write its integrand as $[\exp vx]\,[h(y, v)]$. Now $h(y, v)$ is non-singular in $(-\infty, -k)$ and $0(v^{-3/2})$ independent of y as $v \to -\infty$. It follows e^{vx} and $h(y, v)$ are $L^2(-\infty, -k)$ integrable and we have by Schwarz's inequity

$$\left|\int_{-k}^{-\infty} e^{vx}h(y, v)\, dv\right| \leq \left[\int_{-k}^{-\infty} e^{2xv}\, dv \int_{-k}^{-\infty} |\, h(y, v)\,|^2\, dv\right]^{1/2}$$

$$\leq A\left[\int_{-\infty}^{-k} e^{2xv}\, dv\right]^{1/2} \to 0 \text{ as } x \to \infty \text{ independently of } y.$$

This shows the asymptotic requirements far to the right of the barrier are met.

The Riemann-Lebesque lemma shows the integral in the representations for ϕ when $x < 0$ or when $x > 0$ goes to zero as y becomes negatively infinite. Since this is true of the exponential terms in the case $x > 0$, we see that ϕ decays with depth. Clearly then, all of the order conditions we have had to assume in the formulation of the representations of $\phi(x, y)$ for R_1 and R_2 are satisfied.

VIII. Reflection and Transmission Coeffcients. From the results which we derived in Sec. VI, we may supply the reflection and transmission coefficients of the traveling

surface waves for $y = a$, $x \gtrless 0$. We note that for $x \to \infty$

$$\phi(x, y) = [\gamma_3 \exp(i\sigma x) + \gamma_4 \exp(-i\sigma x)] \exp(\beta y)$$

and this in turn is equal to

$$-\left[\frac{\gamma_1 K_+(\kappa)}{\sigma - \kappa} + \frac{\gamma_2 K_+(-\kappa)}{\sigma + \kappa}\right] \frac{\beta^2}{\sigma} \frac{\exp(\beta)(y - 2a)}{K_+(\sigma)} \exp(i\sigma x)$$

$$+ \left[\frac{\gamma_1 K_+(\kappa)}{-(\sigma + \kappa)} + \frac{\gamma_2 K_+(-\kappa)}{\kappa - \sigma}\right] \frac{\beta^2}{\sigma} \frac{\exp \beta(y - 2a)}{K_+(-\sigma)} \exp(-i\sigma x).$$

From this we have then

$$\gamma_3 = -\left[\frac{\gamma_1 K_+(\kappa)}{\sigma - \kappa} + \frac{\gamma_2 K_+(-\kappa)}{\sigma + \kappa}\right] \frac{\beta^2}{\sigma} \frac{\exp(-2\beta a)}{K_+(\sigma)} \tag{8.1}$$

and

$$\gamma_4 = -\left[\frac{\gamma_1 K_+(\kappa)}{\sigma + \kappa} + \frac{\gamma_2 K_+(-\kappa)}{\sigma - \kappa}\right] \frac{\beta^2}{\sigma} \frac{\exp(-2\beta a)}{K_+(-\sigma)}. \tag{8.2}$$

That is, we have two linear relations between the amplitudes of the traveling waves in the regions $x > 0$ and $x < 0$, $y = a$. The conditions (8.1) and (8.2) are independent as one can readily see by evaluating the determinant of the system (8.1) and (8.2).

As usual we have a relation between the magnitudes of γ_1, γ_2, γ_3 and γ_4. We form the integral

$$\int (\phi\phi_n^* - \phi^*\phi_n)\, ds' \tag{8.3}$$

which is equal to zero since ϕ is source free and sufficiently regular. This integral is evaluated over the "boundary" of the channel with due recognition for the barrier. In view of the particular asymptotic forms we have found for ϕ for $|x| \to \infty$, $v \to -\infty$ (8.3) reduces to

$$|\gamma_3|^2 - |\gamma_4|^2 = [|\gamma_1|^2 - |\gamma_2|^2] \cdot \frac{\beta a \kappa(\beta a - \beta^2 a^2 + \alpha_0^2)[\exp(-2\beta a)][1 + \exp(-2\alpha_0)]}{2\sigma\alpha_0^2(\alpha_0 - a\beta)}. \tag{8.4}$$

The expressions we have found for γ_3 and γ_4 satisfy this relation. On the basis of (8.4) we can define the reflection coefficients and transmission coefficients. Note that if $\gamma_4 = 0$, we have transmission to the right and reflection to the left. $|\gamma_2| = |\gamma_2/\gamma_1|$ is the magnitude of the left reflection coefficient. Then

$$\left|\frac{\gamma_3}{\gamma_1}\right|^2 \frac{2\sigma\alpha_0^2(\alpha_0 - a\beta)\exp(2a\beta)}{\beta a \kappa(\beta a - \beta^2 a^2 + \alpha_0^2)[1 + \exp(-2\alpha_0)]}$$

defines the transmission coefficient to the right, that is $|t_R|$. We find in this case

$$|r_L|^2 = \left(\frac{\sigma - \kappa}{\sigma + \kappa}\right)^2,$$

$$|t_R|^2 = \frac{4\sigma\kappa}{(\sigma + \kappa)^2}.$$

Similarly $| r_R | = | r_L|$ and $| t_L | = | t_R |$, that is for the case in which $\gamma_2 = 0$. Note that for the case $k = 0$

$$| r_L |^2 = | r_R |^2 = \left(\frac{a\beta - \alpha_0}{a\beta + \alpha_0}\right)^2,$$

$$| t_L |^2 = | t_R |^2 = \frac{4a\beta\alpha_0}{(a\beta + \alpha_0)^2}.$$

Had we only been interested in magnitudes of reflection and transmission coefficients, these results might have been anticipated from classical transmission line theory. However, the determination of the various reflection and transmission coefficients involves a phase for each of these coefficients. These angles may be calculated from $K_+(\pi)$, $K_+ (-\kappa)$, $K_+ (\sigma)$ and $K_+ (-\sigma)$ but we shall not pursue this point here.

Bibliography

1. A. E. Heins, *Water Waves over a Channel of Finite Depth with a Dock*, American Journal of Mathematics **LXX**, 730-748 (1948).
2. A. E. Heins, *Water Waves over a Channel of Finite Depth*, Canadian Journal of Mathematics **II**, 210-222 (1950).
3. A. E. Heins, *Some Remarks on the Coupling of Two Ducts*, Journal of Mathematics and Physics, **XXX**, 164-169 (1951).
4. J. J. Stoker, *Surface Waves in Water of Variable Depth*, Q. of Appl. Math., **5**, 1-54 (1947).
5. R. E. A. C. Paley and N. Wiener, *The Fourier Transform in the Complex Domain*, American Mathematical Society Colloquium Publication, 1934, Chapter IV.
6. E. C. Titchmarsh, *Introduction to the Theory of the Fourier Integral*, Oxford 1948, p. 15, Theorem 40.

—NOTES—

THE FUNDAMENTAL THEOREM OF ELECTRICAL NETWORKS*

By J. L. SYNGE (*School of Theoretical Physics, Dublin Institute for Advanced Studies*)

I wish to thank Messrs. L. C. Robbins, Jr. and W. K. Saunders for drawing my attention to what may appear to be a flaw in the reasoning of the paper with the above title.[1] The proof there given is not convincing, and I take this opportunity to supply a fuller proof, in which the reasoning is a little delicate, but which seems to establish the result.

Consider the two propositions:

(A) $$\mathbf{e} = 0 \quad \text{implies} \quad \mathbf{i} = 0.$$

(B) $$\mathbf{Z}' \neq 0.$$

It is stated on p. 127 of the paper cited that (A) implies (B); that is the result we have to establish here.

We have the following equations:

(1) $$\mathbf{i} = \mathbf{C}\mathbf{i}', \qquad \mathbf{e}' = \mathbf{C}_t\mathbf{e}, \qquad \mathbf{e}' = \mathbf{Z}'\mathbf{i}'.$$

These equations embody all our knowledge about the behavior of the network, and any set of vectors $\mathbf{i}$, $\mathbf{i}'$, $\mathbf{e}$, $\mathbf{e}'$ consistent with them are to be regarded as possible. But we must bear in mind the definition of the mesh currents $\mathbf{i}'$ as branch currents (components of $\mathbf{i}$ in branches-out-of-tree); this means that the matrix $\mathbf{C}$ is such that

(2) $$\mathbf{i}' \neq 0 \text{ implies } \mathbf{i} \neq 0.$$

Choose $\mathbf{e} = 0$ and $\mathbf{e}' = 0$. Then (1) read

(3) $$\mathbf{i} = \mathbf{C}\mathbf{i}', \qquad 0 = 0, \qquad 0 = \mathbf{Z}'\mathbf{i}'.$$

Any vectors $\mathbf{i}$, $\mathbf{i}'$ satisfying these equations are possible.

Suppose that (B) is false, i.e. suppose $\mathbf{Z}' = 0$. Then there exists a non-zero $\mathbf{i}'$ satisfying the last of (3), and, by (2), the $\mathbf{i}$ given by the first of (3) is non-zero. Thus, on the assumption that (B) is false, we have a solution of (1) with $\mathbf{e} = 0$ and $\mathbf{i} \neq 0$. Hence, if (B) is false, (A) is false. Therefore if (A) is true, then (B) is true. In other words, (A) implies (B), which is the required result.

---

*Received June 1, 1952.

[1] Q. Appl. Math. 9, 113-127 (1951).

PARALLEL REFLECTION OF LIGHT BY PLANE MIRRORS*

By JOSEPH B. KELLER (*Institute for Mathematics and Mechanics, New York University*)

1. Introduction. If three plane mirrors are mutually orthogonal then, as is well known in geometrical optics, any ray[1] incident on these mirrors in the octant which they bound will be reflected once from each mirror and will emerge parallel to its original direction. Similarly any ray incident on two orthogonal plane mirrors, in the quadrant which they bound and in the plane perpendicular to them, will be reflected once from each mirror and will emerge parallel to its original direction. It is interesting to investigate whether there are other configurations of plane mirrors which also have this property of parallel reflection.

For two mirrors, with the incident ray in the plane perpendicular to them, the answer is easily found although it does not seem to be well known. It is this: if and only if the mirrors meet at an angle π/n, where n is any even integer, every ray will emerge parallel to its original direction. Furthermore every ray suffers n reflections; for example, two in the case of orthogonal mirrors discussed above.

For three mirrors meeting at a point Synge[2] has shown that if every ray is reflected *exactly once* from each mirror every ray will emerge parallel to its original direction if and only if the mirrors are mutually orthogonal. However, in view of our above result for two mirrors, it is reasonable to drop the restriction that every ray be reflected exactly once from each mirror. We then find that there are indeed other configurations of three mirrors meeting at a point which yield parallel reflection for every incident ray. These are the configurations in which the sets of dihedral angles are $(\pi/2, \pi/3, \pi/4)$, $(\pi/2, \pi/3, \pi/5)$ and $(\pi/2, \pi/2, \pi/n)$ where n is an even integer. The numbers of reflections suffered by any ray are 9, 15 and $n + 1$ respectively. The case $(\pi/2, \pi/2, \pi/n)$ with $n = 2$ is the case of mutually orthogonal mirrors, in which case each ray is reflected three times. Furthermore, these are the only configurations of three or more mirrors meeting at a point which have the parallel reflection property.

In section II we formulate the problem, and in section III we deduce the result described above.

2. Formulation. Suppose three planes meet at a single point and that one side of each plane is designated as the reflecting side. Then one and only one of the eight regions into which the planes divide space will be on the reflecting side of all three mirrors. Any oriented straight line in this region which can be extended indefinitely in its negative direction without intersecting a plane is called an incident ray[1]. If an incident ray intersects a plane it is reflected according to the law of reflection; the reflected ray may be again reflected, etc. If a reflected ray can be extended indefinitely in its positive direction

---

*Received June 2, 1952. This work was performed at Washington Square College of Arts and Science, New York University and was supported in part by Contract No. AF-19(122)-42 with the U.S. Air Force through sponsorship of Geophysics Research Division, Air Force Cambridge Research Center, Air Materiel Command.

[1]Rays which intersect an edge or vertex or are parallel to one or two mirrors are excluded. The latter rays do emerge parallel to their original directions although they suffer only two or one reflections, respectively, not being reflected from the mirrors to which they are parallel.

[2]J. L. Synge, Reflection in a Corner Formed by Three Plane Mirrors, Quarterly of Applied Math., **4**, 166-176 (1946).

without intersecting a plane, it is called a final reflected ray. The problem we consider is that of finding all configurations of three planes meeting at a point with the property that every final reflected ray is parallel to the incident ray from which it originates.

First, following Synge, we shall reformulate the problem in terms of reflection of points on the surface of a sphere. To this end we construct a sphere with center at the meeting point of the planes. The surface of the sphere intersects the planes in great circles. The trihedral region on the reflecting sides of all three planes cuts out on the sphere a spherical triangle T, which is bounded by the three great circles. We represent a ray by the point of intersection, with the sphere, of a parallel infinite line through the center. Since a straight line through the center intersects the sphere twice the more negative of the two points is chosen. Thus all parallel rays with the same orientation are represented by the same point, while antiparallel rays are represented by antipodal points. It is clear that all incident rays, defined above, are represented by points in the spherical triangle T.

Now suppose that a ray, represented by the point P, is reflected in a plane, represented by the great circle A. Then the reflected ray is represented by a point P_1 which is obtained by reflecting P in A. This can easily be seen if the point of reflection is transported to the center of the sphere. The ray from P to the center is the incident ray; the ray from P_1 to the center is the negative extension of the reflected ray.

Since, by definition, only one side of each plane is reflecting a point P must lie on the reflecting side of a great circle A in order to be reflectable in A. We will call a reflection of P in A admissible if P does lie on the reflecting side of A. The reflecting side of A is the side or hemisphere containing the spherical triangle T, according to the definition of T. Now a ray emerges, or is a final reflected ray in accordance with the definition above, only when it is not reflectable in any plane. Thus a point P represents a final reflected ray only if it lies on the non-reflecting side of each plane. But every point which lies in all three hemispheres not containing T must lie in the spherical Triangle T' antipodal to T. We can now formulate our problem as that of finding all spherical triangles T with the property that every sequence of admissible reflections of any point P in T leads to the antipodal point P' in T', since P' represents a final ray parallel to the incident ray represented by P.

3. Solution. Let us suppose that T is a triangle with the desired property and that P_1 and P_2 are two points in T. We first want to prove that if an admissible sequence of reflections R_1 carries P_1 into some point P, and an admissible sequence of reflections R_2 carries P_2 into the same point P, then P_1 and P_2 are identical. In other words, no two distinct points can have a common image under admissible sequences of reflections. To prove this assertion we note that, by hypothesis, every sequence of admissible reflections of a point of T terminates on the antipodal point. Thus, in particular, the sequence R_1 which carries P_1 into P can be continued by a sequence of admissible reflections which we shall call S_1 so that $R_1 + S_1$ carries P_1 into P_1', the antipodal point. But then S_1 carries P into P_1'. Therefore $R_2 + S_1$ carries P_2 through P into P_1'. Since this is an admissible sequence of reflections, it must by hypothesis terminate on P_2'. Therefore P_1' is identical with P_2' and so P_1 is identical with P_2.

Now let us consider the neighborhood of a vertex of T of vertex angle α. We wish to consider successive reflections of a point of T in the two sides meeting at the vertex. We introduce polar coordinates with the vertex as origin and one side as polar axis. If ϕ is the polar angle the other side is given by $\phi = \alpha$. If $\phi = \theta$ is the angular coordinate

of a point of T, its image after n successive reflections in the sides 0 and α will have the coordinate $\phi_n(\theta)$ given by:

$$\left.\begin{array}{ll}\phi_n(\theta) = \begin{array}{ll} n\alpha + \theta & n \text{ even} \\ -(n-1)\alpha - \theta & n \text{ odd} \end{array} & \text{if the first reflection is in side } 0, \\ \\ \phi_n(\theta) = \begin{array}{ll} (n-1)\alpha - \theta & n \text{ odd} \\ -n\alpha + \theta & n \text{ even} \end{array} & \text{if the first reflection is in side } \alpha. \end{array}\right\} \tag{1}$$

These reflections are admissible only until the image falls in the vertical angle π to $\pi + \alpha$ (or $-\pi$ to $-\pi + \alpha$) since then the image is on the non-reflecting side of both mirrors meeting at the vertex. [To prove the result stated previously for two mirrors, we set $\phi_n = \pi + \theta$ or $-\pi + \theta$ and find $\alpha = \pi/2$ with n even.]

From the expressions in Eq. 1 it is not difficult to show that *some* image of every point will lie in the vertical angle π to $\pi + \alpha$. Let us consider the least value of n for which the image of any point of the original angle lies in the vertical angle. The sequence of admissible reflections which carries a particular point into the vertical angle in this number of reflections is necessarily an admissible sequence for all points in the original angle. Therefore the image of the original angular region under this sequence is another angular region which overlaps the vertical angle. If this overlapping is complete, then one may deduce, as above, that $\alpha = \pi/n$ where n is an odd or even integer. If the overlapping is incomplete the image covers an edge of the vertical angle. The points outside this edge may then be reflected in it, and some of them will fall on already covered points. Therefore distinct points of the original angle will have a common image, contradicting the theorem above. Thus the overlapping must be complete and therefore, we conclude that each vertical angle is of the form π/n, where n is an integer, not necessarily the same for all vertices.

Combining this result with the fact that the sum of the interior angles of a spherical triangle exceed π, we obtain the inequality.

$$\frac{\pi}{n_1} + \frac{\pi}{n_2} + \frac{\pi}{n_3} > \pi. \tag{2}$$

The only solutions of this inequality are:

$$\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{3}\right), \quad \left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}\right), \quad \left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{5}\right), \quad \left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{n}\right). \tag{3}$$

Thus the only possible triangles having the desired property are included here since these solutions were obtained by imposing necessary conditions.

By examining the admissible reflections of these triangles on a sphere, we find that only the following actually do yield parallel reflection:

$$\left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}\right), \quad \left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{5}\right), \quad \left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{n}\right), n \text{ even}. \tag{4}$$

The other triangles do map onto their antipodal triangles but with the sides interchanged; in other words, a point does not land on its antipode but on the antipode of its image in the principal altitude of the triangle. In the same process we find that the respective

numbers of reflections are 9, 15 and $n + 1$. This completes the proof for the case of three mirrors.

For more than three mirrors all the considerations preceding equation 2 are valid, but the corresponding inequality has no integer solutions at all.

To prove only Synge's result, which is included above, a much simpler method suffices. If a point arbitrarily near a vertex is reflected successively once in each of the sides meeting there the image remains arbitrarily close to the vertex. The third reflection must bring it to the antipode, an angular distance arbitrarily close to π radius. Thus the angular distance from the vertex to the third side is arbitrarily near $\pi/2$, and this applies to each side. Thus the triangle must be an octant of a sphere, which completes the necessity proof. The sufficiency is proved by reflecting an octant three times in its sides.

It is of interest to compare our results with those described on pps. 81–83 of Coxeter's "Regular Polytopes" (Methuen, London, 1948). There the discrete groups of reflections generated by three or more concurrent planes are deduced. It is first found that all the dihedral angles must be of the form π/n, then equation 2 is applied, and the solutions are given by equation 3, with none for more than three planes. In the problem of discrete groups all sequences of reflections are permitted, while in the present problem only "admissible" sequences of reflections are permitted, which is a weaker condition. On the other hand we have the condition that some admissible sequence must carry each point into its antipode, and this condition is strong enough to limit the solutions to those given in equation 4—a subset of the configurations generating discrete groups.

The new parallel reflecting configurations given herein may have practical value in some applications because they reflect back only those rays which are incident from a certain angular region, and the angular region may be made much smaller than that of the mutually orthogonal mirrors.

I wish to express my indebtedness to my colleague, Prof. Wilhelm Magnus, who supplied a crucial part of the necessity proof given above.

NOTE ON SELF-PROPAGATION OF TURBULENT SPOTS*

By CARL E. PEARSON (*Harvard University*)

Introduction. Emmons (1), (2) has recently directed attention towards the manner in which transition occurs from a laminar to a turbulent boundary layer. It appears that the dominating phenomenon is the spontaneous generation of a large number of turbulent "spots" which grow rapidly and eventually coalesce. It has been observed experimentally that, except perhaps immediately after generation, each spot grows in such a manner as to maintain geometric similarity. Because the spot is growing in a boundary layer, the rate at which an elemental portion of its periphery propagates its turbulence outwards is dependent on the orientation of that portion; this dependency of propagation rate on orientation is necessarily associated with the shape of the spot.

It is the purpose of this note to prove the simple but interesting result that for either of two reasonable speculative hypotheses concerning propagation rate, the shape of the spot approaches a certain asymptotic shape which is independent of the initial spot shape.

*Received July 18, 1952.

Material point hypothesis. It is possible to conceive of the propagation of turbulence into a non-turbulent region as being due to the effect of small random eddies being continually induced in the quiescent fluid immediately adjoining the turbulent region. In a laminar boundary layer, the propagation of such induced turbulence may be considered as being at a rate which is dependent on the orientation of the turbulent surface in question and in a direction perpendicular to the surface; this leads us to the following equivalent picture.

Suppose that at every instant of time a set of material points occupy a closed convex curve in the plane. Assume that each material point moves perpendicularly to the curve on which it lies and with a velocity $R(\theta)$ depending only on the slope θ of the curve at that point. Let the family of curves generated in this way be represented parametrically in the complex plane by

$$z(\theta, t) = x(\theta, t) + iy(\theta, t)$$

t being the time. Consider first the rate at which the θ value associated with a particular material point changes in time.

In Fig. (1), let A and B be positions occupied by the curve at times t and $t + \Delta t$,

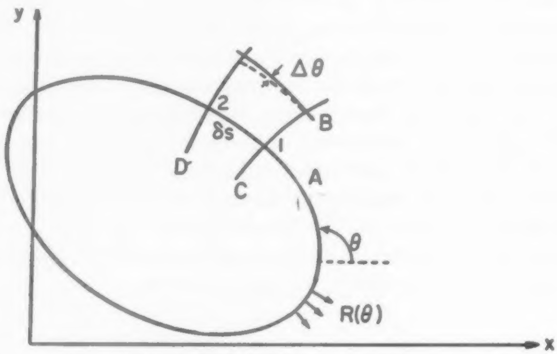

FIG. 1. Neighboring trajectories.

and let C and D be two neighboring orthogonal trajectories (i.e., paths followed by two neighboring particles). Then to the first order, the change in θ for a particle on trajectory D occurring in a time interval Δt is

$$\Delta\theta = -\left(\frac{dR}{d\theta}\,\delta\theta\,\Delta t\right) \Big/ \delta s,$$

where

$$\frac{dR}{d\theta}\,\delta\theta = R(\theta + \delta\theta) - R(\theta),$$

and therefore

$$\frac{d\theta}{dt} = -\frac{dR}{d\theta} \Big/ \frac{ds}{d\theta}.$$

Now the velocity of the material point is

$$\frac{dz}{dt} = \frac{\partial z}{\partial t} + \frac{\partial z}{\partial \theta}\frac{d\theta}{dt}$$

whence

$$\frac{\partial z}{\partial t} = -iRe^{i\theta} + \left(\frac{dR/d\theta}{\partial x/\partial\theta}\cos\theta\right)\left(\frac{\partial x}{\partial\theta} + i\frac{\partial y}{\partial\theta}\right) = e^{i\theta}\left(\frac{dR}{d\theta} - iR\right).$$

But the right hand side is a function of θ alone, and the equation may therefore be integrated to yield

$$z = e^{i\theta}\left(\frac{dR}{d\theta} - iR\right)t + C(\theta),$$

where $C(\theta)$ represents the parametric position of the curve at time $t = 0$. Thus the curve approaches the asymptotic motion

$$x = t\left(R\sin\theta + \frac{dR}{d\theta}\cos\theta\right),$$

$$y = t\left(\frac{dR}{d\theta}\sin\theta - R\cos\theta\right).$$

(These two equations may also be used to determine R is the asymptotic shape is known from experiment).

Wavelet hypothesis. Each point on the periphery of a turbulent spot may be regarded as the continual origin of a disturbance which spreads outwards from it in time; the instantaneous envelope of these disturbances is then the self-propagating curve itself. The shape of the disturbance surrounding any point of origin need not be circular if the region into which it spreads is inhomogeneous (as in the case of a laminar boundary layer). Let the velocity of the (infinitesimal) disturbance be R (β) as shown in Fig. (2).

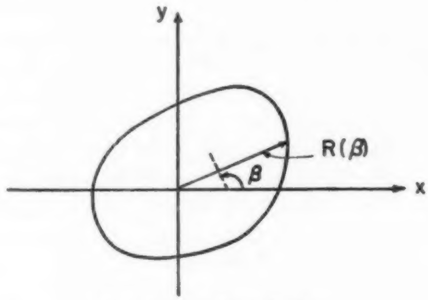

FIG. 2. Wavelet velocity.

Then if the equation of the closed curve is

$$x = f(\theta, t), \qquad y = g(\theta, t),$$

the equation of a wavelet emanating from the point (θ) will be, after time Δt,

$$x - f(\theta, t) = \Delta t R(\beta)\sin\beta,$$

$$y - g(\theta, t) = \Delta t R(\beta)\cos\beta.$$

The equation of a neighboring wavelet will be

$$x - f(\theta + \Delta\theta, t) = \Delta t R(\beta + \Delta\beta) \sin(\beta + \Delta\beta),$$
$$y - g(\theta + \Delta\theta, t) = -\Delta t R(\beta + \Delta\beta) \cos(\beta + \Delta\beta),$$

and hence the relation between θ and β is obtained from

$$\frac{\partial f}{\partial \theta} \Delta\theta = -\Delta t \frac{\partial}{\partial \beta}(R \sin \beta)\, \Delta\beta,$$
$$\frac{\partial g}{\partial \theta} \Delta\theta = \Delta t \frac{\partial}{\partial \beta}(R \cos \beta)\, \Delta\beta,$$

whence

$$\frac{\partial g/\partial\theta}{\partial f/\partial\theta} = \tan\theta = -\frac{\partial(R\cos\beta)/\partial\beta}{\partial(R\sin\beta)/\partial\beta}. \tag{1}$$

This equation gives the direction β of the intersection point of the wavelets as a function of θ.

Now consider again the curve to be composed of material points, this time moving in the β-direction of Eq. (1).

$$\begin{aligned}\frac{d}{dt}(\tan\theta) &= \lim_{\Delta\theta\to 0,\, \Delta t\to 0} \frac{1}{\Delta t}\left\{\frac{y(\theta+\Delta\theta, t+\Delta t) - y(\theta, t+\Delta t)}{x(\theta+\Delta\theta, t+\Delta t) - x(\theta, t+\Delta t)} - \frac{y(\theta+\Delta\theta, t) - y(\theta, t)}{x(\theta+\Delta\theta, t) - x(\theta, t)}\right\} \\ &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\{\frac{\partial y/\partial\theta - \Delta t(d\beta/d\theta)\,\partial(R\cos\beta)/\partial\beta}{\partial x/\partial\theta + \Delta t(d\beta/d\theta)\,\partial(R\sin\beta)/\partial\beta} - \frac{\partial y/\partial\theta}{\partial x/\partial\theta}\right\} \\ &= -\frac{\partial y/\partial\theta}{\partial x/\partial\theta}\left\{\frac{(d\beta/d\theta)\,\partial(R\cos\beta)/\partial\beta}{\partial y/\partial\theta} + \frac{(d\beta/d\theta)\,\partial(R\sin\beta)/\partial\beta}{\partial x/\partial\theta}\right\} \\ &= -\frac{d\beta/d\theta}{\partial x/\partial\theta}\left\{\frac{\partial}{\partial\beta}(R\cos\beta) + \tan\theta\frac{\partial}{\partial\beta}(R\sin\beta)\right\} = 0,\end{aligned}$$

by (1). Then

$$\frac{dx}{dt} = R\sin\beta = \frac{\partial x}{\partial t} + \frac{\partial x}{\partial\beta}\frac{d\beta}{dt} = \frac{\partial x}{\partial t}$$

whence

$$x = (R\sin\beta)t + F(\beta).$$

and similarly

$$y = -(R\cos\beta)t + G(\beta),$$

where $F(\beta)$ and $G(\beta)$ are the parametric representations of the original curve, β being now used as the parameter for convenience. Thus there is an asymptotic shape, in fact, that of Fig. (2).

References

(1) H. W. Emmons, *The laminar-turbulent transition in a boundary layer*, Part I, J. Aero. Sci. **18** 490 (1951).

(2) H. W. Emmons, Part II, to appear in the Proc. of 1st U. S. National Congress. Theor. Appl. Mech.

ON THE DEVELOPMENT OF PLASTIC HINGES IN RIGID-PLASTIC BEAMS*

By M. G. SALVADORI and F. DIMAGGIO (*Columbia University*)

Synopsis. A study is made of the development of plastic hinges in free-free, rigid-plastic beams acted upon by certain types of distributed, dynamic loads, characterized by a *concentration parameter c*, whose value varies between zero and infinity as the load distribution varies from a uniform to a concentrated load.

It is found that as the load intensity increases a plastic hinge is first developed at the center of the beam, whatever the value of the concentration parameter.

As the load intensity increases beyond this value, the development of successive hinges depends upon the parameter c: two additional hinges develop laterally if $c > 3.45$, while the central hinge first splits if $c < 3.45$ before lateral hinges can develop. In the first case the lateral hinges wander toward the center of the beam. In the second a wider central portion of the beam becomes plastic.

1. Introduction. The elasto-plastic behavior of free-free beams under dynamic loads or initial velocities has recently been studied by Bleich and Salvadori[1], who obtained expressions for the elasto-plastic displacements and the plastic deformations in terms of infinite series expansions of normal modes.

Bleich and Salvadori showed that upper bounds for the plastic angle developed at the center of a free-free beam under symmetrical loads or velocities may be obtained by considering the beam as a rigid-plastic body (as suggested by Prager for concentrated dynamic loads). These bounds are good approximations of the true plastic angle whenever the dynamic loads or initial velocities are so high as to produce bending moments several times larger than the capacity moment. Another approximation of the plastic angle may be obtained by freezing the beam into a rigid-plastic body at the time when the first hinge is developed. In the case of initial velocities this approximation is often a lower bound of the plastic angle.

Lee and Symonds[2] obtained upper bounds for the plastic angle and studied the successive development of plastic hinges in rigid-plastic, free-free beams acted upon by a normal concentrated dynamic load P at the middle point. They proved that, after the appearance of a central plastic hinge, two additional hinges develop laterally, which, as the load increases, wander toward the center of the beam.

In what follows the development of plastic hinges due to a symmetrical distributed load p is studied by means of a *concentration parameter c*, which permits the distribution of p to vary from a uniform load p_0, corresponding to $c = 0$, to a concentrated load P, corresponding to $c = \infty$.

2. Rigid-plastic beams. A rigid-plastic beam is an idealized beam of infinite rigidity capable of becoming plastic suddenly at a section where the bending moment due to the loads and the inertia forces reaches a given value, the so-called *capacity value* M_0. The material of a rigid-plastic beam has therefore a stress-strain diagram of the type indi-

*Received June 10, 1952. This paper presents results of research conducted at Columbia University under contract Nonr-266 (08) with the Office of Naval Research.

[1]H. H. Bleich and M. G. Salvadori, *Vibration analysis of elasto-plastic structures, with application to impulsive motion*, Office of Naval Research Project NR-360-002, Contract Nonr-266(08), Technical Report No. 1, (1952).

[2]E. H. Lee and P. S. Symonds, *Large plastic deformations of beams under transverse impact*, J. Appl. Mech. 19, 308-314 (1952).

cated in Fig. 1. Let us consider, in particular, a rigid-plastic, free-free beam of length $2l$ and constant mass m per unit of length, acted upon by a distributed normal load $p(x, t)$ and referred to an x-axis with origin at its middle point (Fig. 2). Under the assumption that the load p be symmetrical with respect to the origin and maximum at the origin, the bending moment will at first be maximum at 0, since the average pressure $P/(2l)$ and the constant inertia forces per unit of length are equal and opposite, and the moment of the pressure $p(z) - P/(2l)$ is maximum at 0 (Fig. 3). In this first phase

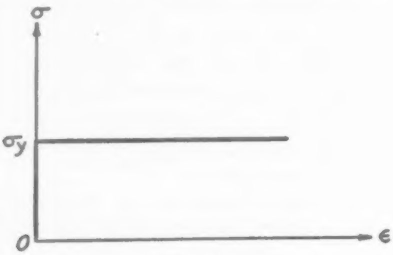

Fig. 1.

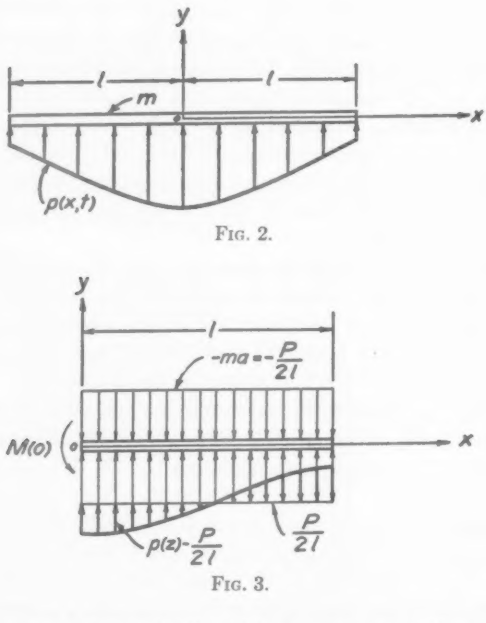

Fig. 2.

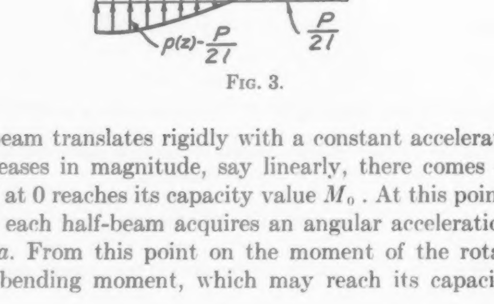

Fig. 3.

of the motion the beam translates rigidly with a constant acceleration a in the y direction and, as p increases in magnitude, say linearly, there comes a moment when the bending moment M at 0 reaches its capacity value M_0. At this point a hinge is suddenly developed at 0 and each half-beam acquires an angular acceleration $-\alpha$ on top of the linear acceleration a. From this point on the moment of the rotational inertia forces contributes to the bending moment, which may reach its capacity value at sections

other than $x = 0$. An analysis of this second phase of the motion, limited to the development of additional lateral hinges or to the splitting of the central hinge, is contained in what follows.

3. The concentration parameter. The type of load considered is distributed symmetrically about the origin according to the exponential law:

$$p(z) = p_c \frac{c}{2} (1 + cz)e^{-cz} \qquad (0 \leqq z \leqq 1) \tag{1}$$

where:

$$z = x/l \tag{2}$$

and c is a constant called the *concentration parameter*.

The total load on the beam due to the distribution (1) equals

$$\begin{aligned} P &= p_c l \frac{c}{2} 2 \int_0^1 (1 + cz)e^{-cz} \, dz \\ &= 2p_c l\left(1 - e^{-c} - \frac{c}{2} e^{-c}\right) \end{aligned} \tag{3}$$

and the pressure p_c is so chosen to make P independent of c, so that

$$p_c = \frac{P/(2l)}{1 - e^{-c}(1 + c/2)} \cdot \tag{4}$$

It is seen from Eqs. (1) and (4) that as c approaches zero $p(z)$ approaches a uniform distribution $p_0 = P/(2l)$, while as c approaches infinity $p(z)$ becomes a concentrated load P applied at $x = 0$. Figure 4 gives the graphs of $p(z)$ for $c = 0, 2, 6$ and 10.

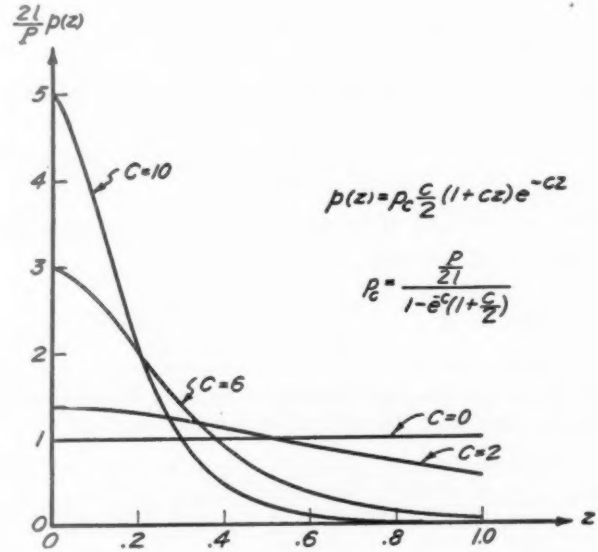

FIG. 4.

4. The rigid translation phase. The acceleration a of the beam in the y-direction, before the bending moment reaches its capacity value at 0, is given by:

$$a = \frac{P}{2ml} = (p_c/m)[1 - e^{-c}(1 + c/2)]. \tag{5}$$

The corresponding inertia forces $-ma$ and the load $p(z)$, per unit length, create a moment $M(0)$ at the origin evaluated by the rotational equilibrium of a half-beam about 0 (Fig. 5):

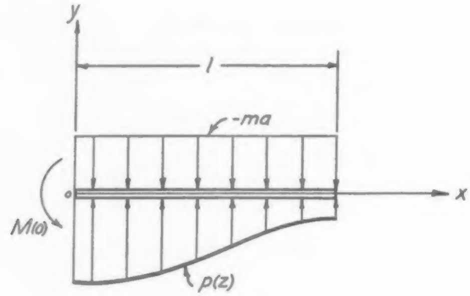

FIG. 5.

$$M(0) = -p_c l^2 \frac{c}{2} \int_0^1 (1 + cz)ze^{-cz}\, dz + p\, \frac{l^2}{2}\left[1 - e^{-c}\left(1 + \frac{c}{2}\right)\right]$$

$$= \frac{p_c l^2}{2}\,[1 - 3/c + e^{-c}(2 + c/2 + 3/c)] \tag{6}$$

It is convenient to introduce a non-dimensional *load parameter* μ defined by the equation

$$\mu = \frac{Pl}{M_0}, \tag{7}$$

where P is given by Eq. (3) and M_0 is the plastic capacity moment of the beam. In terms of μ the moment $M(0)$ becomes:

$$M(0) = M_0\mu\, \frac{1 - 3/c + e^{-c}(2 + c/2 + 3/c)}{4[1 - e^{-c}(1 + c/2)]}\,. \tag{8}$$

A central hinge develops and the rigid translation phase of the motion ends when $M(0)$ reaches the value M_0, that is for the value μ_0 of μ defined by:

$$\mu_0 = 4\,\frac{1 - e^{-c}(1 + c/2)}{1 - 3/c + e^{-c}(2 + c/2 + 3/c)}\,. \tag{9}$$

It is seen from Eq. (9) that the value μ_0 of μ at which the central hinge appears approaches

4 as c approaches infinity (concentrated load). As c approaches zero μ_0 becomes indeterminate, but expansion of e^{-c} into a power series gives the asymptotic value

$$\mu_0 \doteq \frac{48}{c^2} - 8, \qquad (c \ll 1) \tag{10}$$

which approaches infinity as c approaches zero. The graph of μ_0 versus c appears in Fig. 6a.

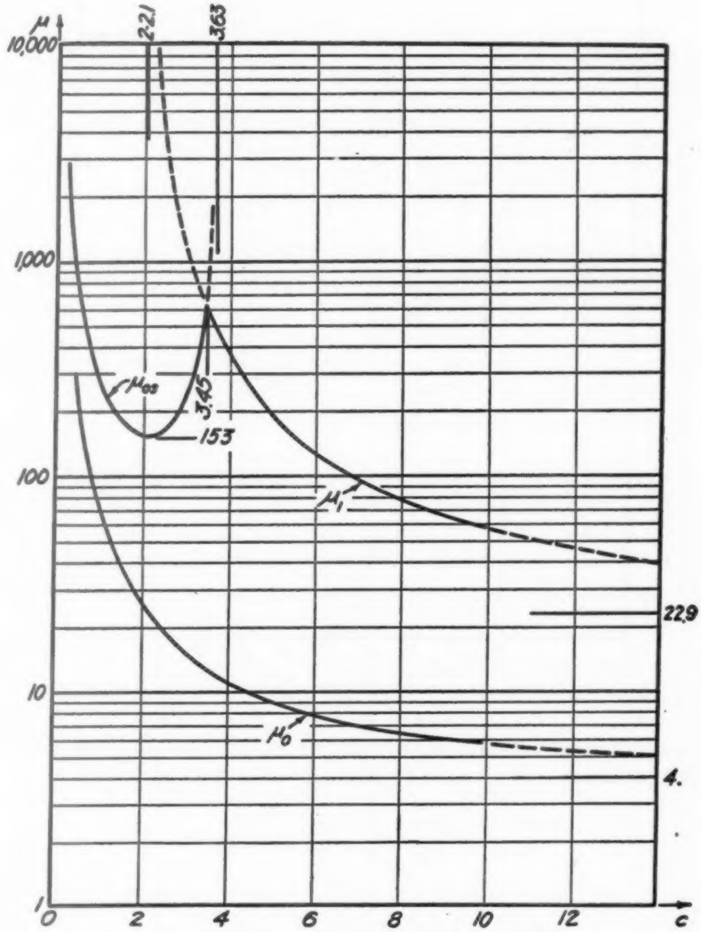

FIG. 6a.

5. The translational-rotational phase. For values of μ larger than μ_0 the half-beams translate with an acceleration a and rotate rigidly with an angular acceleration $-\alpha$, so that the acceleration of the section $x = lz$ of the beam becomes:

$$a(z) = a - \alpha l z. \tag{11}$$

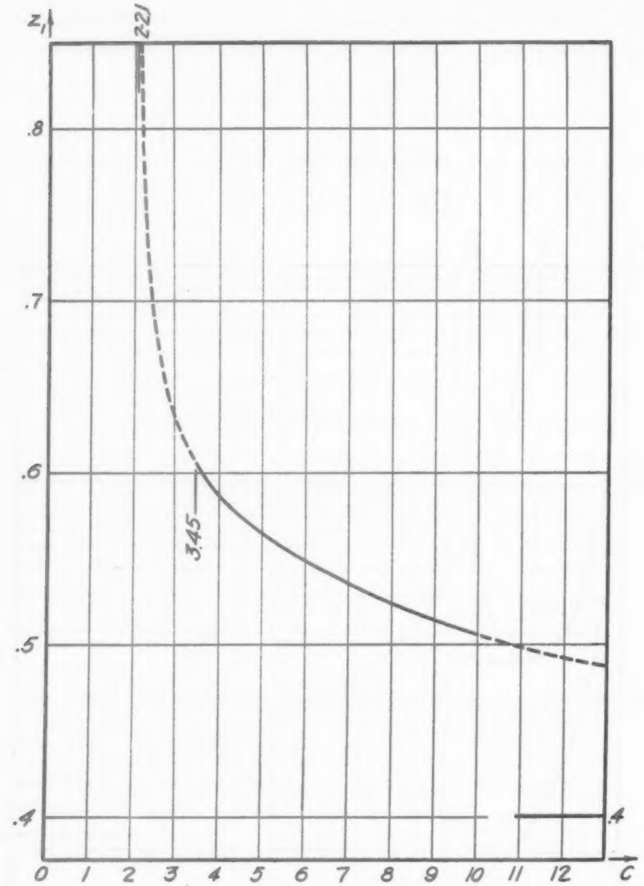

Fig. 6b.

The two unknown accelerations a and αl are determined by applying the equations of dynamics in translation and rotation to the half-beam (Fig. 7):

$$P/2 - \int_0^1 ma(z)\, dz = 0 \tag{12a}$$

$$M_0 + \int_0^1 p(z)z\, dz - m \int_0^1 a(z)z\, dz = 0. \tag{12b}$$

By means of Eqs. (11), (1) and (3) these equations reduce to:

$$ma - \tfrac{1}{2}m\alpha l = p_c[1 - e^{-c}(1 + c/2)]$$

$$\tfrac{1}{2}ma - \tfrac{1}{3}m\alpha l = M_0/l^2 + \tfrac{1}{2}p_c[3/c - e^{-c}(3 + 3/c + c)]$$

and give the following values of ma and $m\alpha l$:

$$ma = -6M_0/l^2 + p_c[4 - 9/c + e^{-c}(5 + 9/c + c)] \tag{13a}$$

$$m\alpha l = -12M_0/l^2 + p_c[6 - 18/c + e^{-c}(12 + 18/c + 3c)] \tag{13b}$$

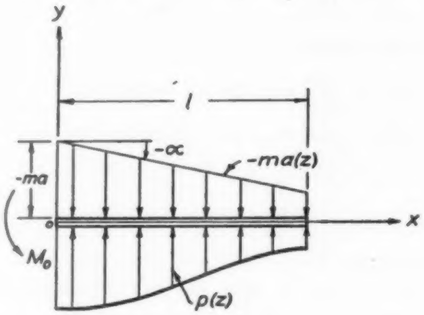

FIG. 7.

By means of Eqs. (1), (7), (11) and (13) the bending moment at any section $x = lz$ becomes (calling u the dummy variable of integration)

$$\begin{aligned} M(z) &= l^2 \int_1^z [p(u) - ma(u)](u - z)\,du \\ &= \frac{M_0}{2[1 - e^{-c}(1 + c/2)]} \Big\{ 2[1 - e^{-c}(1 + c/2)](1 - 3z^2 + 2z^3) \\ &\qquad + \frac{\mu}{2}\{(3/c - 2z) - e^{-cz}(z + 3/c) \\ &\qquad + z^2[(4 - 9/c) + e^{-c}(5 + 9/c + c)] \\ &\qquad - z^3[2 - 6/c + e^{-c}(4 + 6/c + c)]\} \Big\}. \end{aligned} \tag{14}$$

A second hinge will develop in the beam as soon as $M(z)$ has a stationary value at, say $z = z_1 \neq 0$, and this stationary value equals the capacity moment M_0 in absolute value. Since $M(z)$ is certainly maximum at $z = 0$, even after the hinge has developed at 0, because of symmetry, the stationary value of M at z_1 must be a minimum (Fig. 8)

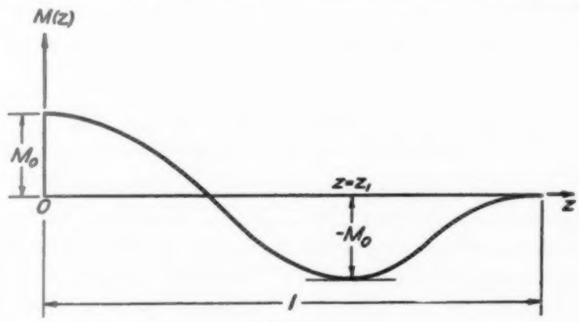

FIG. 8.

or $M(z)$ would have a maximum larger than M_0 at another section. Hence the section z_1, at which a second hinge develops as μ increases beyond μ_0 to a value μ_1, is defined by the conditions

$$\frac{M(z)}{M_0} = -1 \qquad \frac{\partial M}{\partial z} = 0, \tag{15}$$

which by means of Eq. (14) become:

$$2[1 - e^{-c}(1 + c/2)](2 - 3z^2 + 2z^3) + \mu/2\{(3/c - 2z) - e^{-cz}(z + 3/c) + z^2[(4 - 9/c) + e^{-c}(5 + 9/c + c)] - z^3[2 - 6/c + e^{-c}(4 + 6/c + c)]\} = 0; \tag{16a}$$

$$12[1 - e^{-c}(1 + c/2)](z^2 - z) + \mu\{-[1 - e^{-cz}(1 + \tfrac{1}{2}cz)] + z[(4 - 9/c) + e^{-c}(5 + 9/c + c)] - 3z^2[(1 - 3/c) + e^{-c}(2 + 3/c + c/2)]\} = 0. \tag{16b}$$

Solution of the simultaneous Eqs. (16a) and (16b) for various values of c gives the graphs of μ_1 and z_1 versus c plotted in Figs. 6a and 6b, respectively. It may be shown analytically that as c approaches infinity μ_1 approaches 22.9 and z_1 approaches 0.4; while, as c decreases, the second hinge moves outward and μ_1 increases, approaching infinity as c approaches a value between 2.0 and 2.1.

6. The splitting of the central hinge. The values of z_1 and μ_1 obtained above for the development of a second hinge in the half-beam were predicated upon the fact that $M(z)$ would remain a maximum at $z = 0$. This is guaranteed by the equation

$$\left[\frac{\partial^2 M}{\partial z^2}\right]_{z=0} < 0.$$

If instead, as μ increases above μ_0, the second derivative of M becomes positive for values of μ such that

$$\mu_0 < \mu < \mu_1, \tag{17}$$

the moment M would become minimum at 0. Since this cannot happen lest $M(z)$ be greater than M_0 at another section, the central hinge splits into two symmetrical hinges which wander outward plasticizing a central portion of the beam, in which the moment has the constant value M_0. The value $\mu_{0,s}$ of μ at which the central hinge splits is therefore characterized by the condition

$$\left[\frac{\partial^2 M}{\partial z^2}\right]_{z=0} = 0, \tag{18}$$

that is, by the equation:

$$[4 - 9/c - c/2 + e^{-c}(5 + 9/c + c)]\mu + 6[-2 + e^{-c}(2 + c)] = 0,$$

from which:

$$\mu_{0,s} = \frac{6c[2 - e^{-c}(2 + c)]}{(-9 + 4c - c^2/2) + e^{-c}(9 + 5c + c^2)}\,. \tag{19}$$

The graph of $\mu_{0,s}$ versus c, plotted in Fig. 6a, has a minimum for $c = 2.2$, approaches infinity as c approaches zero and 3.63, and *crosses the graph of* μ_1 *for* $c = 3.45$.

It is thus proved that lateral hinges will develop in the beam when $\mu = \mu_1$ while the central hinge remains stationary if $c > 3.45$. If instead $c < 3.45$, the central hinge will split, plasticizing a wider central portion of the beam, when $\mu = \mu_{0,s}$, before lateral hinges may appear.

THE EFFECT OF COALESCENCE IN CERTAIN COLLISION PROCESSES*

By Z. ALEXANDER MELZAK (*McGill University*)

This note is concerned with the variable size-distributions of atmospheric and colloidal particles. The following physical model is adopted: a large number of particles are randomly distributed in space; these particles are in random motion due to e.g. Brownian effect or turbulence; the system of particles is given by the particle-mass distribution. The distribution varies as a result of collision processes in which the randomly moving particles meet and coalesce. The particles do not break up and do not enter or leave the system under consideration. It is assumed that the total number of particles is large enough to justify the use of

(a) the mass density function $f(x, t)$; $f(x, t)dx$ being the average number per unit volume of particles of mass (or volume) x to $x + dx$ at time t,

(b) the coalescence function $\phi(u, v)$, where

$$N(u, v, t) = f(u, t)f(v, t)\phi(u, v)\, du\, dv\, dt, \tag{1}$$

$N(u, v, t)$ being the average number, per unit volume, of collisions resulting in coalescence, between particles of mass u to $u + du$ and v to $v + dv$ respectively, in the time interval t to $t + dt$.

The function $\phi(u, v)$ takes care of such factors as: the collision cross-section, the mobility of the particles, the fraction of collisions resulting in coalescence and any shape factor affecting the collision. With these definitions, the condition for the conservation of mass becomes:

$$\frac{\partial f(x, t)}{\partial t}\, dx\, dt = \frac{1}{2}\int_{u+v=x} N(u, v, t) - \int_{u=0}^{\infty} N(u, x, t). \tag{2}$$

This means that the increase of the number of particles of mass x in time dt is equal to half the number of coalescences between particles of masses u, and $x - u$, $0 \leq u \leq x$, minus the number of coalescences between the particles of mass x and any others. Substituting equation (1) in equation (2) gives the fundamental equation of the process:

$$\frac{\partial f(x, t)}{\partial t} = \frac{1}{2}\int_0^x f(y, t)f(x - y, t)\phi(y, x - y)\, dy - f(x, t)\int_0^\infty f(y, t)\phi(x, y)\, dy. \tag{3}$$

If $N(t)$ is the total number of particles (per unit volume) then:

$$N(t) = \int_0^\infty f(x, t)\, dx, \tag{4}$$

$$\frac{dN(t)}{dt} = -\frac{1}{2}\int_0^\infty \int_0^\infty f(x, t)f(y, t)\phi(x, y)\, dx\, dy. \tag{5}$$

The initial condition will be taken as $f(x, 0) = g(x)$, a known function corresponding to a known initial distribution. If this initial distribution is nearly homogeneous, and if $\phi(x, y)$ does not oscillate rapidly, then for small t equation (5) may be approximated by:

$$\frac{dN}{dt} = -\frac{1}{2}\int_0^\infty \int_0^\infty f(x, t)f(y, t)c\, dx\, dy = -\frac{cN^2}{2}, \qquad c = \text{const.} \tag{6}$$

*Received June 23, 1952. The research reported upon has been sponsored by the Geophysics Research Division of the Air Force Cambridge Research Center under Contract AF 19(122)-217.

This may be regarded as a theoretical foundation of an empirical coagulation law valid for some cases of Brownian motion. Whytlaw-Grey (1932) and Sinclair (1950) discuss the range and validity of equation (6) in more detail.

Of course, equation (6) holds exactly if $\phi(x, y) \equiv c$. In this case, one can solve the main equation (3) under general initial conditions. Letting $2F(x, t) = cf(x, t)$, equation (3) becomes:

$$\frac{\partial F(x, t)}{\partial t} = \int_0^x F(y, t)F(x - y, t)\, dy - 2F(x, t) \int_0^\infty F(y, t)\, dy. \tag{7}$$

One introduces the Laplace Transforms:

$$\psi(p, t) = \int_0^\infty e^{-px}F(x, t)\, dx; \qquad \psi(p, 0) = \frac{c}{2}\int_0^\infty e^{-px}g(x)\, dx. \tag{8}$$

Then, using the standard properties of the Laplace Transforms, one obtains from equation (7) the equation for $\psi(p, t)$:

$$\frac{\partial \psi(p, t)}{\partial t} = \psi^2(p, t) - 2\psi(p, t)\psi(0, t); \qquad \psi(p, 0) \text{ known}. \tag{9}$$

Its solution is:

$$\psi(p, t) = \frac{1}{(Nt + 1)^2} \frac{\psi(p, 0)}{1 - \dfrac{t}{Nt + 1}\psi(p, 0)}; \qquad N = \psi(0, 0) = \frac{CN(0)}{2}. \tag{10}$$

Now, inverting equation (8), one obtains the final solution:

$$f(x, t) = \frac{2}{C(Nt + 1)^2} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{px}\psi(p, 0)}{1 - \dfrac{t}{Nt + 1}\psi(p, 0)}\, dp; \tag{11}$$

$\gamma > 0$ and sufficiently large.

(From equation (11), one may obtain as a particular case a solution of equation (3) given by Schumann (1940), valid for a specific initial distribution only.) The conditions under which the above use of the Laplace Transform is justified are not discussed here, but it is easy to see from the physical situation that they are satisfied.

A few particular results for initial distributions which may be of interest have been worked out. Two of them follow.

If $f(x, 0) = g(x) = A\, \delta(x - 1)$,
then

$$f(x, t) = \frac{2N}{C(Nt + 1)^2} \sum_{j=0}^{\infty} \left(\frac{Nt}{Nt + 1}\right)^j \delta[x - (j + 1)]; \qquad N = \frac{CA}{2}. \tag{12}$$

From equation (12) it follows, as is otherwise clear, that starting with single particles of equal size one will later have double, triple, m-tuple aggregates. The number of m-tuples per unit volume at time t is $\gamma(t)e^{-m\beta(t)}$.

If $g(x) = f(x, 0) = A\, x^n e^{-\alpha x}$, n integer,
then

$$f(x, t) = \frac{2e^{-\alpha x}L}{ct(Nt + 1)(n + 1)} \sum_{j=1}^{n+1} \zeta^j \exp(\zeta^j Lx), \tag{13}$$

where

$$2N = \frac{CAn!}{\alpha^{n+1}}, \qquad L = \left[\frac{cn!tA}{2(Nt + 1)}\right]^{1/n+1}, \qquad \zeta = e^{2\pi i/n+1}.$$

Equation (13) shows on closer analysis that as n increases the oscillatory terms become more and more dominant, giving progressively sharper and higher periodic peaks in $f(: \ t)$.

Equation (3) was also solved under different assumptions for $\phi(x, y)$ viz. $\phi(x, y) = c(x + y)$ and $\phi(x, y) = cxy$. The solutions, except for some special initial conditions, are involved, though quite straightforward. They will not be reproduced here, but may be obtained by transforming equation (3) by means of equations (4) and (5), introducing Laplace Transforms and solving the resulting partial differential equations.

With physically more realistic assumptions for $\phi(x, y)$, e.g. $\phi(x, y) = c(x^{1/3} + y^{1/3})^2$ (capture cross-section for two spheres), equation (3) becomes so involved that it has to be treated numerically.

The following meteorological problem provides an illustration of the theory developed above. Marshall and Palmer (1948) found a simple expression for the distribution of the raindrop sizes. Their exponential relation,

$$N_D = N_0 e^{-\Lambda D}, \tag{14}$$

where $N_D dD$ is the number of drops in the diameter interval $D - D + dD$ in unit value of space,

N_0 is a universal constant,

Λ is a rain-intensity parameter,

fits the observed data of Laws and Parsons (1943) very well except at small diameters. This deviation at small diameters may be explained by assuming the distribution given by equation (14) subject to a collision process with the above form of $\phi(x, y)$. The results are shown in figure 1. An attempt was also made to explain theoretically the whole dis-

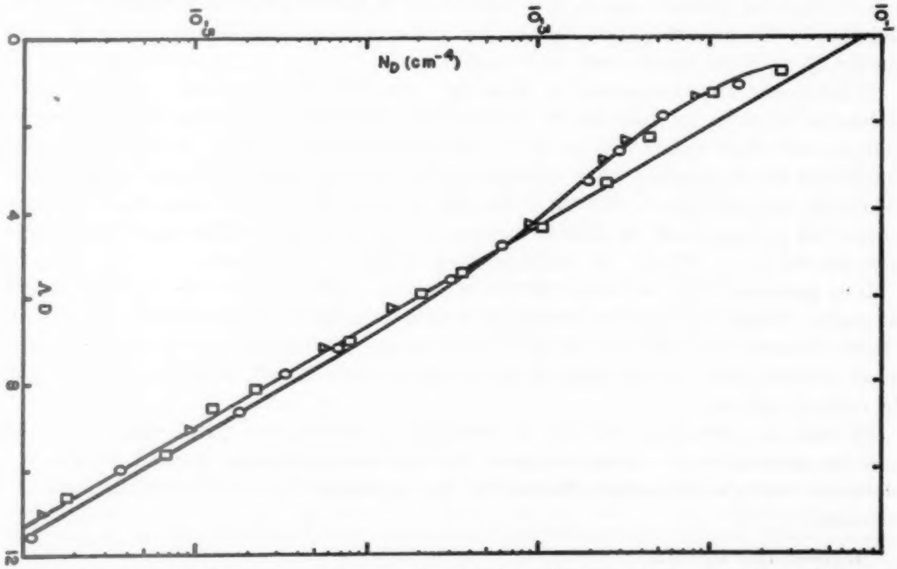

FIG. 1. The marked points represent the experimental data of Laws and Parsons, the straight line is obtained from the Marshall-Palmer relation, the curved line represents the calculated distribution. N_D, Λ, D are explained under Eq. (14) in the text.

tribution given by (14), but the results are not yet satisfactory. This means probably that the form of $\phi(x, y)$, chosen above, does not describe the physical situation sufficiently closely.

The author is indebted to Professor J. S. Marshall, of McGill University, for suggesting the above problem, and to Dr. Walter Hitschfeld, of McGill University, for helpful suggestions concerning the physical model used.

References

J. O. Laws and D. A. Parsons, *The relation of raindrop-size to intensity*, Trans. Amer. Geophys. Union, **24**, part II, 452 (1943).
J. S. Marshall and W. McK. Palmer, *The distribution of raindrops with size*, J. Meteor, **5**, 4, 165 (1948).
T. E. W. Schumann, *Theoretical aspects of the size distribution of fog particles*, Q. J. Roy. Met. Soc., **66** (1940).
D. Sinclair, *Handbook on Aerosols*. Chapter 5, Atomic Energy Commission, Washington (1950).
Whytlaw-Grey, *Smoke*. E. Arnold and Co., London (1932).

VARIATION OF COEFFICIENTS OF SIMULTANEOUS LINEAR EQUATIONS[1]

By JOHN E. BROCK (*Midwest Piping Supply Co., Inc.*)

1. Introduction. In dealing with sets of simultaneous linear equations it sometimes becomes necessary to modify one or more of the coefficients (to correct errors or for other reasons) after the solution has been obtained. Methods of obtaining the corresponding modifications in the solution with a minimum amount of computation have been treated by B. L. Weiner[2], and in a discussion of Weiner's paper (which bears the same title as the present paper), I. F. Morrison[3] indicates a matrix formulation of the analysis. In the present paper, we develop three particular prodedures each of which appears to be quite simple and quite useful.

Matrices are used throughout in these developments. Both matrices and scalars will be denoted by ordinary italic letters. Letters with subscripts will denote matrix elements (scalars) and other scalars will be easily distinguished from matrices. A symbol denoting a matrix or matrix product when enclosed within parentheses and followed by subscripts will denote the appropriate element of the matrix, thus$(ab)_{ij} = c_{ij}$ where $ab = c$. Superscripts will be used both to denote exponents and to serve as distinguishing indices; there should be no difficulty in distinguishing between these usages.

It is presumed that we know the set of inverse coefficients for the system, that is, the matrix which is inverse to (reciprocal to) the original coefficient matrix. This is not a severe demand since the reciprocal matrix is frequently already known or can be computed without much trouble making use of calculations already performed in obtaining the original solution.

So that the operations will not be obscured by voluminous calculations, the order, n, of the systems in the examples chosen for illustrative purposes is small. However, it is obvious that the advantage afforded by the procedures described here increases as n increases.

[1]Received June 15, 1952.

[2]B. L. Weiner, *Variation of coefficients of simultaneous linear equations*, Trans. ASCE, **113**, 1349 (1948) (Paper No. 2358).

[3]I. F. Morrison, ibid., p. 1379.

2. Symmetrical Systems. Suppose a is a symmetrical, non-singular matrix and that we know $\alpha = a^{-1}$, its inverse, and the solution, x, a column matrix, to the equation

$$ax = c. \tag{1}$$

Now let a be changed to

$$a^* = a + b, \tag{2}$$

where all elements of b are zero except only

$$b_{pq} = b_{qp} = \beta; \tag{3}$$

that is, the equal and symmetrically disposed elements a_{pq} and a_{qp} are increased by the amount β. Our problem is to find a new solution

$$x^* = x + y \tag{4}$$

such that

$$a^* x^* = c. \tag{5}$$

In the derivation,[4] let the matrices b^{rs} be defined by

$$(b^{rs})_{ij} = \delta_i^r \delta_j^s \tag{6}$$

where the (Kronecker) deltas are zero unless the upper and lower indices are the same in which case their value is unity. In establishing the relation (14) below, we use the "summation convention" which implies a summation from 1 to n inclusive over all repeated indices except p and q which are fixed. Then

$$(\alpha b^{rs})_{ij} = \alpha_{ir} \delta_j^s , \tag{7}$$

$$(b^{mn} \alpha b^{rs})_{ij} = \alpha_{nr} \delta_i^m \delta_j^s , \tag{8}$$

$$b = \beta(b^{pq} + b^{qp}), \tag{9}$$

$$(\alpha b)_{ij} = \beta(\alpha_{ip} \delta_j^q + \alpha_{iq} \delta_j^p), \tag{10}$$

$$(b\alpha b)_{ij} = \beta^2(\alpha_{qp} \delta_i^p \delta_j^q + \alpha_{qq} \delta_i^p \delta_j^p + \alpha_{pp} \delta_i^q \delta_j^q + \alpha_{pq} \delta_i^q \delta_j^p), \tag{11}$$

$$\begin{aligned}(\alpha b \alpha b)_{ij} &= \beta^2(\alpha_{ip}\alpha_{qp} \delta_j^q + \alpha_{ip}\alpha_{qq} \delta_j^p + \alpha_{iq}\alpha_{pp} \delta_j^q + \alpha_{iq}\alpha_{pq} \delta_j^p) \\ &= \beta\alpha_{pq}(\alpha b)_{ij} + \beta^2(\alpha_{ip}\alpha_{qq} \delta_j^p + \alpha_{iq}\alpha_{pp} \delta_j^q),\end{aligned} \tag{12}$$

$$\begin{aligned}(b\alpha b \alpha b)_{ij} &= \beta\alpha_{pq}(b\alpha b)_{ij} + \beta^3(\alpha_{qp}\alpha_{qq} \delta_i^p \delta_j^p + \alpha_{pp}\alpha_{qq} \delta_i^p \delta_j^q \\ &\qquad + \alpha_{pp}\alpha_{qq} \delta_i^q \delta_j^p + \alpha_{pq}\alpha_{pp} \delta_i^q \delta_j^q) \\ &= 2\beta\alpha_{pq}(b\alpha b)_{ij} + \beta^2(\alpha_{pp}\alpha_{qq} - \alpha_{pq}^2) b_{ij} .\end{aligned} \tag{13}$$

---

[4]The reviewer has suggested adding at this point a remark concerning motivation. The general case, Section 4, came first to the writer's attention. Success in summing the series (37) in particular cases leads to the analyses presented in Sections 2 and 3. Essentially one attempts to determine powers of the matrix $m = \alpha b$, and in Section 2 is led to Equation (14) and in Section 3 to Equation (31); having these results, the summation is easily accomplished. However, in presenting the results in these two Sections (2 and 3), it is not necessary to exhibit the series. Once Equations (14) and (31) are established, the desired results follow simply by a little algebraic manipulation. Use of index notation, the Kronecker deltas, and the summation convention abbreviates the derivation of Equations (14) and (31).

Thus we have established the relation

$$bmm = Abm + Bb, \tag{14}$$

where

$$m = \alpha b = a^{-1}b, \tag{15}$$

$$A = 2\beta\alpha_{pq}, \tag{16}$$

$$B = \beta^2(\alpha_{pp}\alpha_{qq} - \alpha_{pq}^2). \tag{17}$$

We now prove the important formulas

$$y = \frac{m(mx) - (1 + A)(mx)}{1 + A - B}, \tag{18}$$

and

$$x^* = x + y = x + \frac{m(mx) - (1 + A)(mx)}{1 + A - B}. \tag{19}$$

To do this, form the product a^*x^* and in simplifying note that

$$am = aa^{-1}b = b. \tag{20}$$

We have

$$\begin{aligned} a^*x^* &= (a + b)(x + y) = ax + bx + (a + b)y \\ &= c + \left\{b + \frac{a + b}{1 + A - B}[mm - (1 + A)m]\right\}x \\ &= c + [(1 + A)b - Bb + bm + bmm - (1 + A)b - bm - Abm]\frac{x}{1 + A - B} \\ &= c + (bmm - Abm - Bb)\frac{x}{1 + A - B} = c, \end{aligned} \tag{21}$$

making use of Equation (14). Q. E. D. (Of course we require that $B \neq 1 + A$.)

We may also easily obtain the corrected inverse $\alpha^* = (a^*)^{-1}$. Write Equation (19) in the form

$$x^* = \gamma x, \tag{22}$$

where

$$\gamma = I + \frac{mm - (1 + A)m}{1 + A - B}. \tag{23}$$

If we postmultiply this matrix in succession by the columns of α we obtain the corrected columns, i.e., the columns of α^*. In other words

$$\alpha^* = \gamma\alpha$$

Example:

$$a = \begin{bmatrix} 19 & 10 & 4 \\ 10 & 5 & 2 \\ 4 & 2 & 1 \end{bmatrix}; \quad \alpha = \begin{bmatrix} -1 & 2 & 0 \\ 2 & -3 & -2 \\ 0 & -2 & 5 \end{bmatrix}; \quad x = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}; \quad c = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}. \tag{24}$$

Now suppose

$$a^* = \begin{bmatrix} 19 & 10 & 4 \\ 10 & 5 & 0 \\ 4 & 0 & 1 \end{bmatrix}. \qquad \text{Then} \qquad b = (-2)\cdot\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \text{and}$$

we calculate: $\beta = -2$, $p = 2$, $q = 3$, $A = (2)(-2)(-2) = 8$,

$$B = (-2)^2[(-3)(5) - (-2)^2] = -76; \qquad 1 + A - B = 85, \qquad 1 + A = 9,$$

$$m = \begin{bmatrix} 0 & 0 & -4 \\ 0 & 4 & 6 \\ 0 & -10 & 4 \end{bmatrix}; \qquad mx = \begin{bmatrix} 4 \\ -18 \\ 26 \end{bmatrix};$$

$$y = \frac{[m - (1 + A)I](mx)}{1 + A - B} = \begin{bmatrix} -9 & 0 & -4 \\ 0 & -5 & 6 \\ 0 & -10 & -5 \end{bmatrix}\begin{bmatrix} 4 \\ -18 \\ 26 \end{bmatrix} \div 85 = \begin{bmatrix} -140 \\ 246 \\ 50 \end{bmatrix} \div 85;$$

$$x^* = x + y = \begin{bmatrix} 30 \\ -9 \\ -35 \end{bmatrix} \div 85.$$

This is the easiest arrangement of the calculations. If the new inverse matrix is required, as would be the case, for example, if it were necessary to apply a second set of corrections, we would compute:

$$\gamma = \begin{bmatrix} 85 & 40 & 20 \\ 0 & 5 & -6 \\ 0 & 10 & 5 \end{bmatrix} \div 85;$$

$$\alpha^* = \gamma\alpha = \begin{bmatrix} -5 & 10 & 20 \\ 10 & -3 & -40 \\ 20 & -40 & 5 \end{bmatrix} \div 85; \qquad x^* = \gamma x = \begin{bmatrix} 30 \\ -9 \\ -35 \end{bmatrix} \div 85.$$

3. Single Corrections in Non-Symmetrical Systems. Suppose again that α, x, and C satisfy Equation (1), that a is modified as in Equation (2), and that we desire to find x^* given in Equation (4) so as to satisfy Equation (5). However, now we do not require that a be symmetrical and we presume b has only one non-vanishing element

$$\beta = b_{pq}.$$

Thus, using a notation introduced earlier, we can write

$$b = \beta b^{pq}, \tag{26}$$

$$m_{ij} = (\alpha b)_{ij} = \beta \alpha_{ip} \delta_j^q , \tag{27}$$

$$m_{qq} = \beta \alpha_{qp} , \tag{28}$$

$$m_{qq} b_{ij} = \beta^2 \alpha_{qp} \delta_i^p \delta_j^q , \tag{29}$$

$$(bm)_{ij} = \beta^2 \alpha_{qp} \delta_i^p \delta_j^q . \tag{30}$$

Thus, we have proved that

$$bm_{qq} = bm. \tag{31}$$

Now we argue that

$$y = \frac{-mx}{1 + m_{qq}}, \tag{32}$$

$$x^* = \gamma x, \tag{33}$$

where

$$\gamma = I - \frac{m}{1 + m_{qq}}, \tag{34}$$

for

$$\begin{aligned} a^* x^* &= ax + bx - \frac{(a + b)m}{1 + m_{qq}} \cdot x \\ &= c + [b + bm_{qq} - am - bm] \cdot \frac{x}{1 + m_{qq}} = c, \end{aligned} \tag{35}$$

making use of Equations (20) and (31). Also, as before

$$\alpha^* = \gamma \alpha. \tag{36}$$

Example:

$$a = \begin{bmatrix} 1 & 4 & 1 & 3 \\ 0 & -1 & 3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & -2 & 5 & 1 \end{bmatrix}; \quad \alpha = \begin{bmatrix} -5 & 15 & 19 & -8 \\ 9 & 17 & 1 & -12 \\ 4 & 10 & -2 & 2 \\ 3 & -31 & -7 & 18 \end{bmatrix} \div 44; \quad x = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}; \quad c = \begin{bmatrix} 2 \\ 6 \\ 4 \\ 14 \end{bmatrix}.$$

Suppose now that $a_{31} = 3$ is changed to 5. Then $p = 3$, $q = 1$, $\beta = 2$,

$$m = \alpha b = \begin{bmatrix} 19 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -7 & 0 & 0 & 0 \end{bmatrix} \div 22; \qquad m_{qq} = \frac{19}{22}$$

$$\gamma = I - \frac{44m}{63} = \begin{bmatrix} 22 & 0 & 0 & 0 \\ -1 & 41 & 0 & 0 \\ 2 & 0 & 41 & 0 \\ 7 & 0 & 0 & 41 \end{bmatrix} \div 41$$

$$x^* = \gamma x = \begin{bmatrix} 22 \\ -42 \\ 84 \\ 48 \end{bmatrix} \div 41; \qquad \alpha^* = \gamma\alpha = \begin{bmatrix} -5 & 15 & 19 & -8 \\ 17 & 31 & 1 & -22 \\ 7 & 20 & -2 & 3 \\ 4 & -53 & -7 & 31 \end{bmatrix} \div 82.$$

4. General Case. Both the preceding procedures which give exact corrections in convenient finite form were originally motivated by the analysis which follows and evolved from it through success in actually summing the matrix series which appears below.

Suppose again that a, x, and c satisfy Equation (1), that a is modified as in Equation (2), and that we desire to find x^* given in Equation (4) so that Equation (5) is satisfied. We now make no restriction on b other than that the infinite matrix series

$$\gamma = I - m + m^2 - m^3 + \cdots \tag{37}$$

converge, and indeed in such a manner that its terms may be rearranged.

Then, as before, Equations (22) and (24) hold, for

$$a^*x^* = (a + b)\gamma x = c + (-a + a\gamma + b\gamma)x = c, \tag{38}$$

since the quantity in parentheses may be written as

$$\begin{aligned}(-a + a\gamma + b\gamma) &= -a + a - am + am^2 - am^3 + am^4 - \cdots \\ &\qquad\qquad + b - bm + bm^2 - bm^3 + \cdots \\ &= (b - am)(I - m + m^2 - m^3 + \cdots) = (b - am)\gamma,\end{aligned} \tag{39}$$

by reference to Equation (20).

In practice, to investigate the convergence and then to sum a sufficient number of terms of the series to assure the desired accuracy may prove prohibitively laborious. Therefore, it is suggested that the procedure implied in Equations (37), (22), and (24) be used only if it is clearly obvious that the corrections b are so small relative to a that not only is convergence assured but also adequate accuracy may be obtained from the first few terms of the series.

Example:

$$a = \begin{bmatrix} 36 & 16 & 4 \\ 15 & 9 & 3 \\ 6 & 4 & 2 \end{bmatrix}; \quad \alpha = \begin{bmatrix} 3 & -8 & 6 \\ -6 & 24 & -24 \\ 3 & -24 & 42 \end{bmatrix} \div 24; \quad x = \begin{bmatrix} 17 \\ -50 \\ 57 \end{bmatrix} \div 8; \quad c = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix};$$

$$\alpha^* = \begin{bmatrix} 36.02 & 16 & 4 \\ 15 & 9 & 2.99 \\ 6 & 4 & 2 \end{bmatrix}; \quad b = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \div 100;$$

$$m = \alpha b = \begin{bmatrix} 3 & 0 & 4 \\ -6 & 0 & -12 \\ 3 & 0 & -12 \end{bmatrix} \div 1200 = \begin{bmatrix} .002500 & 0 & .003333 \\ -.005000 & 0 & -.010000 \\ .002500 & 0 & -.010000 \end{bmatrix};$$

$$m^2 = \begin{bmatrix} 21 & 0 & 60 \\ -54 & 0 & -168 \\ 45 & 0 & 156 \end{bmatrix} \div (1200)^2 = \begin{bmatrix} .000015 & 0 & .000042 \\ -.000037 & 0 & -.000117 \\ .000031 & 0 & .000108 \end{bmatrix};$$

$$\gamma \approx I - m + m^2 = \begin{bmatrix} .997515 & 0 & -.003291 \\ .004963 & 1 & .009883 \\ -.002469 & 0 & .990108 \end{bmatrix};$$

$$x^* = \gamma x \approx \begin{bmatrix} 2.096271 \\ -6.169037 \\ 7.049273 \end{bmatrix}.$$

The new inverse matrix α^* may be calculated from Equation (24). A complete new solution gives the values

$$x^* = \begin{bmatrix} 2.096277 \\ -6.169052 \\ 7.049277 \end{bmatrix}$$

and it is seen that good accuracy is obtained with our method using only the first three terms of the series given in Equation (37).

ON A SOLUTION OF THE ENERGY EQUATION FOR A ROTATING PLATE STARTED IMPULSIVELY FROM REST*

By RONALD F. PROBSTEIN (*Princeton University*)

1. Introduction. The problem of the steady, laminar incompressible flow of a fluid over an infinite plate rotating at a constant velocity was first solved by von Kármán [1]. Recently the associated heat transfer problem for the von Kármán example was

*Received Nov. 3, 1952. This work has been supported by the United States Air Force, Air Research and Development Command, under Contract AF 33(038)-250.

calculated by Millsaps and Pohlhausen [2]. Neglecting heat conductivity Nigam [3] has obtained the corresponding non-steady velocity distributions valid for small times for the rotating plate started impulsively from rest with a constant angular velocity. It is the purpose of the present paper to give the temperature distributions associated with the velocity fields found by Nigam. Apart from an academic viewpoint this solution offers an interesting mathematical analogy to the steady hypersonic laminar flow over a semi-infinite flat plate with a "self-induced"[1] pressure gradient.

Under the assumption of an incompressible fluid a solution of the energy equation will be obtained for a thermally insulated surface and for a non-insulated surface with a wall temperature distribution which is a parabolic function of the radius. By analogy with Pohlhausen's steady, laminar boundary layer analysis [5], the assumption of constant density requires that the azimuthal plate velocity must be small compared with the sound speed. Furthermore, in the non-insulated case as the viscous dissipation is to be included, it is implied that the energy dissipation associated with viscosity is of the same order as the energy associated with the imposed temperature difference. The maximum permissible difference between the wall and ambient temperature must therefore be small, that is, of the order of the square of the azimuthal plate velocity. Both of these conditions place a limit on the radius beyond which the solutions to be obtained are no longer valid.

Like Nigam's solution of the equations of motion, the present solution of the energy equation will neglect all time dependent terms of order t^2 or higher; that is, only the early stages of the motion when t is small will be considered. Such a procedure is tantamount to an asymptotic expansion in which the functions are expressed in ascending power series in t and the first coefficients is obtained. This of course has the limitation that no information is given regarding the time to reach the steady state. (See Ref. 6 for an analogous procedure in the case of the impulsively started cylinder.).

2. Basic equations. The basic partial differential equations for the time dependent motion ($\partial/\partial t \neq 0$) of an incompressible fluid (density ρ = constant) are most suitably expressed for this problem in cylindrical coordinates r, θ, z with radial, azimuthal and axial velocity components u, v, w. Since the angular velocity of the plate Ω will be assumed constant for all time $t > 0$ the problem is axially-symmetric and hence the partial derivatives with respect to θ vanish. Therefore the energy equation is written as

$$c_v\left\{\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + w\frac{\partial T}{\partial z}\right\} = \frac{\lambda}{\rho}\left\{\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right\}$$
$$+ \nu\left\{2\left(\frac{\partial u}{\partial r}\right)^2 + 2\left(\frac{u}{r}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)^2\right\}, \qquad (1)$$

where, T is the temperature of the fluid, c_v the specific heat expressed in mechanical units, λ the thermal conductivity, and ν the kinematic viscosity.

Nigam has shown that if the infinite plate at time $t = 0$ is suddenly made to rotate with a constant angular spin Ω about the axis $r = 0$, that the velocity components may

---

[1]The term "self-induced" refers to the pressure gradient generated as a result of the curvature of the boundary layer over a flat plate; this pressure gradient in turn affects the boundary layer growth [4].

be expressed as follows:

$$u = \Omega^2 r t f(\eta), \tag{2}$$

$$v = \Omega r g(\eta), \tag{3}$$

$$w = -4\nu^{1/2}\Omega^2 t^{3/2} h(\eta), \tag{4}$$

where, $\eta = z/2\ (\nu t)^{\frac{1}{2}}$ and the functions $g(\eta)$ and $f(\eta)$ satisfy the second order ordinary linear differential equations

$$g'' + 2\eta g' = 0, \tag{5}$$

$$f'' + 2\eta f' - 4f = -4g^2. \tag{6}$$

Note of course, that as the isothermal Blasius velocity distribution remained unaffected by heat transfer in the Pohlhausen analysis [5], so here, the solutions for the velocity components are unchanged by the heat transfer.

It is now assumed that the energy equation is satisfied by a relation of the form

$$c_v T = c_v T_\infty + (\Omega r)^2 k(\eta) + \Omega^2 \nu t m(\eta), \tag{7}$$

where T_∞ is the ambient temperature. On substituting Eq. (7) along with Eqs. (2)-(4) into the energy equation, neglecting all terms of order t^2 or higher, and equating coefficients of t and r^2, the following two ordinary inhomogenous linear differential equations are obtained for the functions $k(\eta)$ and $m(\eta)$:

$$k'' + 2\eta\sigma k' = -\sigma g'^2, \tag{8}$$

$$m'' + 2\eta\sigma m' - 4\sigma m = -16k, \tag{9}$$

where $\sigma = c_v \nu \rho/\lambda$.[2] The problem will now be particularized to the case of $\sigma = 1$ since an analytic rather than numerical solution is desired for the comparison of any results. The close relation between the above equations with $\sigma = 1$ and Eqs. (5) and (6) satisfied by g and f, is then immediately evident.

3. Solutions of the equations. The solutions for m and k may now be expressed as

$$m = 2[(C + 2)g + (D - 1) - f], \tag{10}$$

$$k = \tfrac{1}{2}[Cg + D - (g - 1)^2], \tag{11}$$

where C and D are constants of integration, the additional two required integration constants A and B being contained in the general solution for f which is given by Nigam's Eq. (10).[3] It is of course to be noted from Eq. (11) that the quantity involving k in the expression for the temperature distribution is a function only of the azimuthal velocity v. If g is required to satisfy the boundary conditions that either at time zero or infinitely far from the plate surface ($\eta = \infty$) the azimuthal velocity $v = 0$, and at the plate surface ($\eta = 0$) for all times greater than zero $v = v_{\text{wall}} = \Omega r$, then g is given by

$$g = 1 - 2\pi^{-1/2} \int_0^\eta e^{-\eta^2}\, d\eta = 1 - \operatorname{erf} \eta = \operatorname{erfc} \eta. \tag{12}$$

---

[2]The Prandtl number $c_p \nu \rho/\lambda$ is equal to $\sigma c_p/c_v$. In the case where $c_p = c_v$, the Prandtl number and σ are of course identical.

[3]The sign in front of the particular solution in Eq. (10) of Ref. 3 should be negative.

The particular solution of the energy equation for the thermally insulated plate is obtained by using the boundary conditions that the temperature gradient normal to the surface must vanish, and infinitely far from the plate surface and at time $t = 0$ the temperature has its ambient value T_∞. The temperature distribution in this case is therefore

$$c_v T = c_v T_\infty + v(v_{\text{wall}} - \tfrac{1}{2}v) + \Omega^2 \nu t 4\{(\pi^{-1/2}e^{-\eta^2} - \eta \operatorname{erfc} \eta)^2 - 2\eta^2 \operatorname{erfc} \eta + (4\pi)^{-1/2}\eta e^{-\eta^2}\} \tag{13}$$

For the case where the wall temperature is a parabolic function of the radius (i.e. $T_{\text{wall}} = \text{constant } r^2 + T_\infty$) the boundary conditions for zero time and infinite distance from the plate are the same as in the insulated problem. On the other hand, the boundary conditions at the plate surface become $m(0) = 0$ and $k(0) = \alpha$, where the dimensionless constant α controls the rate of change of the temperature gradient along the radius. Hence, the solution for the temperature is

$$\begin{aligned} c_v T = c_v T_\infty + v\left\{\left(\frac{1 + 2\alpha}{2}\right)v_{\text{wall}} - \frac{v}{2}\right\} + \Omega^2 \nu t 4\{(\pi^{-1/2}e^{-\eta^2} - \eta \operatorname{erfc} \eta)^2 \\ - (1 + 2\alpha)\eta^2 \operatorname{erfc} \eta - \pi^{-1/2}(1 + 2\eta^2) \operatorname{erfc} \eta \\ + \pi^{-1/2}\eta e^{-\eta^2}[2\pi^{-1} + (1 + 2\alpha)]\}. \end{aligned} \tag{14}$$

The present approach can be extended to wall temperature distributions different from parabolic by use of a method similar to that adopted by Chapman and Rubesin [7] for solving the steady, compressible laminar flow over a semi-infinite flat plate with an arbitrary distribution of surface temperature.

4. Discussion of results. The main point of interest in the solutions of the energy equation represented by Eqs. (13) and (14) is their mathematically analogous nature to the solutions for the flow over a semi-infinite flat plate with a self-induced pressure gradient. (See footnote 1 and Ref. 4). Examination shows that both of the solutions obtained can be considered to be partially composed of a Crocco type integral of the steady flow energy equation for the flow over a flat plate with a Prandtl number equal to unity and a constant surface temperature under the boundary layer approximations (i.e. $T = a + bv + cv^2$). In addition there is what might be called an "induced time" effect represented by the $\Omega^2\nu t$ terms which correspond formally to the self-induced pressure gradient already mentioned.

The portion of the solutions involving only the azimuthal velocity arises from the dissipation term $\nu(\partial v/\partial z)^2$ and the analogue of the constant surface temperature in the steady flow flat-plate case is the parabolic distribution chosen for the impulsively started rotating plate.

By analogy to the work in Ref. 4 the terms in v and $\Omega^2\nu t$ are but the first of an asymptotic expansion in ascending powers of t. Thus the present work for small times bears a closer analogy to the treatment in which the steady hypersonic flow over a semi-infinite flat plate for the region close to the leading edge is considered [8]. In this case solutions are obtained by expansion in powers of $x^{\frac{1}{2}}$, rather than in powers of $x^{-\frac{1}{2}}$ as in Ref. 4 (x is the distance along the plate surface with the leading edge as the origin). The latter solution, which is valid for the region toward the back end of the plate, would in effect correspond to the large-time asymptotic expansion for the present problem.

REFERENCES

1. T. von Kármán, *Über laminare und turbulente Reibung*, Z. angew. Math. Mech. **1**, 244-247 (1921).
2. K. Millsaps and K. Pohlhausen, *Heat transfer by laminar flow from a rotating plate*, J. Aero. Sci. **19**, 120-126 (1952).
3. S. D. Nigam, *Rotation of an infinite plane lamina: boundary layer growth: motion started impulsively from rest*, Q. Appl. Math. **9**, 89-91 (1951).
4. L. Lees and R. F. Probstein, *Hypersonic viscous flow over a flat plate*, Princeton University Aeronautical Engineering Laboratory, Report No. 195, April 20, 1952.
5. E. Pohlhausen, *Der Wärmeaustausch zwischen festen Körpern und Flüssigkeiten mit kleiner Reibung und kleiner Wärmeleitung*, Z. angew. Math. Mech. **1**, 115-121 (1921).
6. S. Goldstein, *Modern developments in fluid dynamics*, vol. I, The Clarendon Press, Oxford, 1938, p. 184.
7. D. R. Chapman and M. W. Rubesin, *Temperature and velocity profiles in the compressible laminar boundary layer with arbitrary distribution of surface temperature*, J. Aero. Sci. **16**, 547-565 (1949).
8. L. Lees, *On the boundary layer equations in hypersonic flow and their approximate solutions*, Princeton University Aeronautical Engineering Laboratory, Report No. 212, September 20, 1952.

ON A CERTAIN ASYMPTOTIC EXPANSION

By A. VAN WIJNGAARDEN (*Mathematical Centre, Amsterdam*)

1. Introduction. This little note is concerned with the asymptotic expansion of a certain function, defined by a series resembling that of the modified Bessel function $I_p(z)$, viz.

$$f(z) = \sum_{k=1}^{\infty} \frac{(z/2)^{2k}}{k!k!k^{1/2}}. \tag{1,1}$$

This function plays a role in a certain physical problem, and values of $f(z)$ for large positive values of the argument had to be determined. In itself the problem is of little general interest but it gives an opportunity to stress again the importance of the use of factorial series and to show the simple means they provide to derive an asymptotic expansion in such a case.

2. The factorial series. The modified Bessel function $I_p(z)$ has the expansion

$$I_p(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k+p}}{k!(k+p)!}. \tag{2,1}$$

As $k!\,k^{1/2} \sim (k+1/2)!$ this series for $p = 1/2$ is, apart from some trivial modifications as multiplication by $(z/2)^{1/2}$ and the addition of a term with $k = 0$, closely related to the series (1, 1).

This relationship is used in the following way. The ratio $(k + 1/2)!/(k!\,k^{1/2})$ can be expanded into a factorial series (cf. Nörlund, Differenzengleichungen):

$$\frac{(k+1/2)!}{k!k^{1/2}} = \sum_{s=0}^{\infty} \frac{(-1)^s\binom{s-1/2}{s}B_s^{(s+1/2)}}{(k+3/2)(k+5/2)\cdots(k+s+1/2)}, \qquad (k > 0), \tag{2,2}$$

where $B_s^{(s+1/2)}$ are generalised Bernoulli numbers. As these alternate in sign with increasing values of s, the terms of the series (2, 2) are all positive. Or,

$$\frac{1}{k!k^{1/2}} = \sum_{s=0}^{\infty} (-)^s\binom{s-1/2}{s}B_s^{(s+1/2)}\frac{1}{(k+s+1/2)!}, \qquad (k > 0), \tag{2,3}$$

Received June 2, 1952.

so that after inserting into (1, 1) and interchanging the order of the summations

$$f(z) = \sum_{s=0}^{\infty} (-)^s \binom{s-1/2}{s} B_s^{(s+1/2)} \sum_{k=1}^{\infty} \frac{(z/2)^{2k}}{k!(k+s+1/2)!}$$
$$= \sum_{s=0}^{\infty} (-)^s \binom{s-1/2}{s} B_s^{(s+1/2)} \left\{ \left(\frac{z}{2}\right)^{-s-1/2} \sum_{k=0}^{\infty} \frac{(z/2)^{2k+s+1/2}}{k!(k+s+1/2)!} - \frac{1}{(s+1/2)!} \right\},$$

or with regard to (2, 1):

$$f(z) = \sum_{s=0}^{\infty} (-)^s \binom{s-1/2}{s} B_s^{(s+1/2)} \left\{ \left(\frac{z}{2}\right)^{-s-1/2} I_{s+1/2}(z) - \frac{1}{(s+1/2)!} \right\}. \qquad (2,4)$$

So far, no approximations have been introduced. It should be noted that the two terms between brackets in (2, 4) may not be separated as the latter term should give rise, according to (2, 3) to a divergent series.

The functions $I_{s+1/2}(z)$ are, as well known, elementary functions. For the first few values of s they are respectively:

$$I_{1/2}(z) = (2/\pi z)^{1/2} \sinh z,$$

$$I_{3/2}(z) = (2/\pi z)^{1/2} \left(\cosh z - \frac{1}{z} \sinh z \right),$$

$$I_{5/2}(z) = (2/\pi z)^{1/2} \left(\sinh z - \frac{3}{z} \cosh z + \frac{3}{z^2} \sinh z \right),$$

$$I_{7/2}(z) = (2/\pi z)^{1/2} \left(\cosh z - \frac{6}{z} \sinh z + \frac{15}{z^2} \cosh z - \frac{15}{z^3} \sinh z \right),$$

$$I_{9/2}(z) = (2/\pi z)^{1/2} \left(\sinh z - \frac{10}{z} \cosh z + \frac{45}{z^2} \sinh z - \frac{105}{z^3} \cosh z + \frac{105}{z^4} \sinh z \right),$$

$$I_{11/2}(z) = (2/\pi z)^{1/2} \left(\cosh z - \frac{15}{z} \sinh z + \frac{105}{z^2} \cosh z - \frac{420}{z^3} \sinh z + \frac{945}{z^4} \cosh z - \frac{945}{z^4} \sinh z \right).$$

Moreover, the numerical values of the coefficients in (2, 4) between the sum-sign and the brackets are for the first few values of s respectively:

s	$(-)^s \binom{s-1/2}{s} B_s^{(s+1/2)}$
0	1
1	3/8
2	65/128
3	1225/1024
4	131691/32768
5	4596669/262144

The series (2, 4) is convergent, and moreover for large values of z manageable.

3. The asymptotic series. For practical purposes, it is however advantageous to give up the convergence of the expansion and turn it into an ordinary divergent asymptotic series with simpler terms to be used for $z \gg 1$. To that end, one puts $\sinh z \sim \cosh z \sim e^z/2$, neglects the second term between the brackets in (2, 4), inserts the formulae given for $I_{s+1/2}(z)$ into (2, 4) and rearranges the terms with respect to decreasing powers of z. Then one obtains

$$f(z) \sim e^z(\pi z)^{-1/2}\left(1 + \frac{3}{4}z^{-1} + \frac{41}{32}z^{-2} + \frac{445}{128}z^{-3} + \frac{26571}{2048}z^{-4} + \frac{505029}{8192}z^{-5} \cdots\right). \quad (3,1)$$

The given terms yield e.g. $f(20) \sim 1.42507 \cdot 10^7$; actually $f(20) = 1.42508 \cdot 10^7$.

A note on my paper

A GENERAL SOLUTION FOR THE RECTANGULAR AIRFOIL IN SUPERSONIC FLOW

Quarterly of Applied Mathematics, XI, 1-8 (1953)

By JOHN W. MILES (*University of California*)

It appears that the application of the Lorentz transformation to unsteady flow problems was suggested previously by P. A. Lagerstrom (unpublished lectures at California Institute of Technology, Pasadena, 1948) and by C. Gardner in his paper *Time-dependant linearized supersonic flow past planar wings*, Comm. on Pure and Appl. Math. **3**, 33-38 (1950).

BOOK REVIEWS

Electronic analog computers. By G. A. Korn and T. M. Korn, McGraw-Hill Book Company, Inc., New York, Toronto, London, xv + 378 pp. $7.00.

This book is both a text and an engineering reference book on the subject of d.c. electronic analog computors. The type of computer under consideration is that employed by many commercial and government institutions. These computers are used in the solution of such problems as are encountered in the design of feedback control systems and in the development of new aircraft configurations.

In the book, the problems associated with the use of computers are first discussed. Examples are given of setup procedures and applications to show both the limitations and the versatility of these devices. Following this, the principal engineering design considerations associated with the computer components are discussed. The d.c. amplifiers employed are analyzed at length as are the various means of obtaining special functional representations. The book is concluded with a description of several of the commercially available computers.

As a text on electronic analog computers, it is unfortunate that the a.c. computers are not considered as well as the d.c. computers. Because of this the book is not sufficiently balanced for a text on analog computers and would have to be supplemented by additional reading. The principal value of the book is for the orientation of mathematicians and others concerned with the solution of dynamic problems on analog computers that have either been specially designed or commercially procured.

H. T. Marcy

THE FOURTH SYMPOSIUM ON PLASTICITY

will be held at Brown University, Providence, Rhode Island on September 1, 2 and 3, 1953. The three principal topics of this Symposium will be:

> **Stress-strain relations** (perfectly plastic, work-hardening, visco-elastic, and visco-plastic materials)
>
> **Distribution of stress and strain in bodies exhibiting inelastic properties** (static and dynamic problems, infinitesimal and finite deformations)
>
> **Structural analysis and design in the inelastic range** (limit analysis, static and dynamic loading, shke-a down, inelastic buckling).

Both foreign and American experts in the field will present papers covering their most recent experimental and analytical research, and everyone is invited to participate in the discussion.

Certain sessions of the Symposium, containing papers of particular interest to civil engineers, are being sponsored, jointly with Brown University, by the Engineering Mechanics Division of the American Society of Civil Engineers. The remaining sessions are under the joint sponsorship of the Office of Naval Research and Brown University.

Persons interested in receiving further information concerning the Symposium should write to *Professor H. J. Weiss, Graduate Division of Applied Mathematics, Brown University, Providence 12, R. I.* Copies of the program and reservation cards will be sent out approximately six weeks before the date of the Symposium.

CONTENTS—*continued from cover 4*

THE SECOND U. S. NATIONAL CONGRESS OF APPLIED MECHANICS

Plans have now matured to hold the Second National Congress of Applied Mechanics at the University of Michigan, Ann Arbor, Michigan during June 14-18, 1954. The National Congresses, of which the first was held at Chicago, Illinois, are held every four years between the International Congresses and are planned to supplement these rather than compete with them. The next International Congress will be held at Brussels in 1956. No attempt will be made to attract participation by workers outside the U. S. and Canada in the National Congress, although there will be no rule against such participation.

All workers in the field are cordially invited to submit papers for the Second National Congress. Papers should constitute original research in Applied Mechanics, which includes kinematics, dynamics, vibrations, and waves; mechanical properties of materials and failure; stress analysis, elasticity, and plasticity; fluid mechanics; thermodynamics. Their length must not exceed 5,000 words or the equivalent in equations, tables, and diagrams. It is intended that all papers accepted by the Editorial Committee, with the advice of recognized authorities in the respective field of interest, will be published in full in bound and printed Proceedings of the Congress. It is hoped that these Proceedings will appear within the year following the Congress.

To be considered for presentation at the Congress, complete papers and abstracts must be submitted to the Chairman of the Editorial Committee before January 1, 1954, and such presentation is subject to the full manuscript having been accepted prior to May 15, 1954. Consideration of a paper for publication in the Proceedings can be based only upon the full manuscript, and only papers actually presented at the Congress are eligible for publication.

The papers will be grouped by subject and one half hour will be allotted for presentation and discussion of each paper. Arrangements will also be made for general lectures by outstanding authorities on subjects of general interest to members of the Congress. Get-togethers will be provided for exchange of ideas and social contact.

Inquiries regarding the Congress should be addressed to the Secretary of the Congress, Professor Jesse Ormondroyd, Dept. of Engg. Mechanics, University of Michigan.

CONTENTS

CONTENTS—*continued on cover 3*